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ON FINITE-DIFFERENCE APPROXIMATIONS FOR NORMALIZED BELLMAN EQUATIONS

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Abstract. A class of stochastic optimal control problems involving optimal stopping is considered. Methods of Krylov [15] are adapted to investigate the numerical solutions of the corresponding normalized Bellman equations and to estimate the rate of convergence of finite difference approximations for the optimal reward functions.

1. Introduction

Stochastic optimal control and optimal stopping problems have many applications in mathematical finance, portfolio optimization, economics and statistics (sequential analysis). Optimal stopping problems can be in some cases solved analytically [20]. With most problems, one must resort to numerical approximations of the solutions. One approach is to use controlled Markov chains as approximations to controlled diffusion processes, see e.g. [19]. A thorough account of this approach is available in [18].

We are interested in the rate of convergence of finite difference approximations to the payoff function of optimal stopping and control problems. Using the method of randomized stopping (see [10]) such problems can be treated as optimal control problems with the reward and discounting functions unbounded in the control parameter. This leads us to approximating a normalized degenerate Bellman equation.

Until quite recently, there were no results on the rate of convergence of finite difference schemes for degenerate Bellman equations. A major breakthrough is achieved by Krylov in [11] for Bellman equations with constant coefficients, followed by rate of convergence estimates for Bellman equations with variable coefficients in [12] and [13]. The estimate from [13] is improved in [2] and [1]. Finally, Krylov [14] (published in [15]) establishes the rate of convergence $\tau^{1/4} + h^{1/2}$ of finite difference schemes to degenerate Bellman equations with Lipschitz coefficients given on the whole space, where τ and h are the mesh sizes in time and space respectively. This is later extended to finite difference approximations of Bellman equations on cylindrical domains in [4].

In the present paper we extend this estimate to cover normalized degenerate Bellman equations corresponding to optimal stopping of controlled diffusion processes with variable coefficients. Adapting ideas and techniques of [14] we obtain the rate of convergence $\tau^{1/4}+h^{1/2}$, as in [14]. There are two key ideas which are already introduced in [11] -[13]. The first idea is that the original equation and its approximation should play symmetric roles.

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The other idea is to 'shake' the original equation and its approximation, and to mollify the solutions of the 'shaken equations' to obtain smooth supersolutions to the original equation and to its approximation, respectively, which are close to their true solutions. To implement these ideas one needs appropriate estimates on the regularity of the solutions to the original equation and to its approximation. The necessary regularity estimates on the optimal reward functions, i.e., the solutions of the Bellman equations are well-known, see [10]. Namely, under general conditions the optimal reward funtions are Lipschitz continuous in the space variable and they are Hölder continuous, with exponent $1/2$, in the time variable. The main problem is to obtain the corresponding regularity estimates for the finite difference approximations. In [15] a discrete gradient estimate in the space variable is proved for the solutions to finite difference schemes for degenerate Bellman equations. Hence not only the Lipschitz continuity in the space variable of the finite difference approximations follows but a suitable estimate on their time regularity as well.

Our first main task in the present paper is to extend the discrete gradient estimate from [15] to the case of finite difference schemes for normalized Bellman equations. This is Theorem 4.1 below. We note that in [17] a more general estimate is proved. From Theorem 4.1 the Lipscitz continuity in the space variable of the finite difference approximations follows easily. However, due to the normalizing factor in the finite difference scheme, Theorem 4.1 does not imply the estimate we need on the time regularity of the finite difference approximations. In fact, the time regularity of the solutions does not hold in general, unless we assume stronger conditions on the finite difference scheme than those of Theorem 4.1. Since our main concern in the present paper is the rate of convergence of finite difference approximations for the reward function of optimal stopping of controlled diffusion processes, we establish the necessary time regularity estimate only for these approximations. This is Theorem 6.4, which is the discrete counterpart of Theorem 6.2 on the Hölder continuity in time of the optimal reward function. Hence, using also the regularity of the optimal reward functions and the maximal principle for normalized Bellman equations and for their 'monotone approximations', we prove our rate of convergence estimate, Theorem 2.4 by a straitforward adaptation of the method of 'shaking and smoothing' from [15].

Rate of converge results for optimal stopping are proved for general consistent approximation schemes in [7]. However, the rate $\tau^{1/4} + h^{1/2}$ is obtained only when the diffusion coefficients are independent of the time and space variables. For further results on numerical approximations for Bellman equations we refer to [8], [9] and [3].

The paper is organized as follows. The main result, Theorem 2.4 is formulated in the next section. In Section 3 the existence and uniqueness of solutions to finite difference schemes, Theorem 3.4, is proved together with a result on comparison of the solutions, Lemma 3.9. The gradient estimate on the solutions of finite difference schemes is proved in Section 4, together with important corollaries. An estimate on Lipschitz continuity in the space variable for the reward functions and a result on comparison of the reward functions with supersolutions to Bellman equations are presented in Section 5. The estimate on Hölder continuity in time of the reward functions together with the corresponding estimates for their finite difference approximations are given in Section 6. Theorem 2.4 is proved in Section 7.

2. The Main Result

Fix $T \in (0,\infty)$, and set $H_T = [0,T) \times \mathbb{R}^d$ and $\bar{H}_T = [0,T] \times \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, carrying a d' dimensional Wiener martingale $W = (W_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Below we introduce some basic notions and notation of the theory of controlled diffusion processes from [10]. The notation $|a| = (\sum_{i,j} a_{ij}^2)^{1/2}$, $|b| = (\sum_i b_i)^{1/2}$ and $c^+ = c_+ = (|c| + c)/2$, $c^- = c_- = (-c)_+$ is used for matrices $a \in \mathbb{R}^{k\times l}$, vectors $b \in \mathbb{R}^k$ and real numbers c. Unless otherwise stated, the summation convention with respect to repeated indices is in force throughout the paper.

Let A be a separable metric space and let $\sigma = \sigma^\alpha(t,x)$, and $\beta = \beta^\alpha(t,x)$ be given Borel functions of $(\alpha, t, x) \in A \times \mathbb{R} \times \mathbb{R}^d$, taking values in $\mathbb{R}^{d\times d'}$ and $\mathbb{R}^d$, respectively. Assume that $A = \cup_{n=1}^\infty A_n$ for an increasing sequence of Borel sets A_n of A such that the following assumption holds.

Assumption 2.1. For every integer $n \geq 1$ there is a constant K_n such that for all $\alpha \in A_n$

$$|\sigma^\alpha(t,x) - \sigma^\alpha(t,y)| + |\beta^\alpha(t,x) - \beta^\alpha(t,y)| \leq K_n|x-y| \tag{2.1}$$

$$|\sigma^\alpha(t,x)| + |\beta^\alpha(t,x)| \leq K_n(1+|x|) \tag{2.2}$$

for all $(t,x) \in \bar{H}_T$.

A progressively measurable process $\alpha = (\alpha_t)_{t\geq 0}$ with values in A is called an *(admissible) strategy* if there is an integer $n \geq 1$ such that $\alpha_t(\omega) \in A_n$ for all $t \geq 0$ and $\omega \in \Omega$. The set of strategies with values in A_n is denoted by $\mathfrak{A}_n$, and so $\mathfrak{A} = \bigcup_{n=1}^\infty \mathfrak{A}_n$ is the set of all strategies. By the classical existence and uniqueness theorem of Itô, Assumption 2.1 ensures that for each $\alpha \in \mathfrak{A}$, $s \in [0,T]$ and $x \in \mathbb{R}^d$ there is a unique solution $x^{\alpha,s,x} = \{x_t : t \in [0, T-s]\}$ of

$$x_t = x + \int_0^t \sigma^{\alpha_u}(s+u, x_u)dW_u + \int_0^t \beta^{\alpha_u}(s+u, x_u)du. \tag{2.3}$$

Let $f = f^\alpha(t,x)$ and $c = c^\alpha(t,x)$ be Borel functions of $(\alpha,t,x) \in A\times\mathbb{R}\times\mathbb{R}^d$ with values in $\mathbb{R}$ and $\mathbb{R}_+$, respectively, and let $g = g(t,x)$ be a Borel function of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ with values in $\mathbb{R}$ such that the following assumption holds.

Assumption 2.2. The function g is continuous and there are some constants K and $q \geq 0$ such that

$$|g(t,x)| \leq K(1+|x|^q) \quad \text{for all } (t,x) \in \bar{H}_T. \tag{2.4}$$

For every integer $n \geq 1$ there are constants K_n and $q_n \geq 0$ such that for all $\alpha \in A_n$

$$c^\alpha(t,x) \leq K_n(1+|x|^{q_n}), \quad |f^\alpha(t,x)| \leq K_n(1+|x|^{q_n}) \tag{2.5}$$

for all $(t,x) \in \bar{H}_T$.

For $s \in [0,T]$ we use the notation $\mathfrak{T}(T-s)$ for the set of stopping times $\tau \le T-s$. Consider the following *optimal reward functions*:

$$v(s,x) = \sup_{\alpha \in \mathfrak{A}} v^{\alpha}, \quad (s,x) \in \bar{H}_T, \tag{2.6}$$

$$w(s,x) = \sup_{\alpha \in \mathfrak{A}} \sup_{\tau \in \mathfrak{T}(T-s)} w^{\alpha,\tau}(s,x), \quad (s,x) \in \bar{H}_T, \tag{2.7}$$

where

$$v^{\alpha}(s,x) = \mathbb{E}^{\alpha}_{s,x}\left[\int_0^{T-s} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + g(T,x_{T-s})e^{-\varphi_{T-s}}\right], \tag{2.8}$$

$$w^{\alpha,\tau}(s,x) = \mathbb{E}^{\alpha}_{s,x}\left[\int_0^{\tau} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + g(s+\tau,x_\tau)e^{-\varphi_\tau}\right], \tag{2.9}$$

$$\varphi_t = \varphi_t^{\alpha,s,x} = \int_0^t c^{\alpha_r}(s+r,x_r^{\alpha,s,x})dr,$$

and $\mathbb{E}^{\alpha}_{s,x}$ denotes the expectation of the expression behind it, with $x_t^{\alpha,s,x}$ in place of x_t everywhere. We call v and w the optimal reward functions for the *optimal control* problem, and for the *optimal control and stopping* problem, respectively, with strategies from $\mathfrak{A}$, under *utility rate* f, *terminal utility* g and *discount rate* c. It is useful to notice that for

$$v_n(s,x) = \sup_{\alpha \in \mathfrak{A}_n} v^{\alpha}(s,x), \quad w_n(s,x) := \sup_{\alpha \in \mathfrak{A}_n} \sup_{\tau \in \mathfrak{T}(T-s)} w^{\alpha,\tau}(s,x)$$

we have $v_n(s,x) \uparrow v(s,x)$ and $w_n(s,x) \uparrow w(s,x)$ as $n \to \infty$. Our aim is to investigate finite difference approximations for a class of nonlinear PDEs, called *normalized Bellman PDEs*, to approximate w via finite difference schemes for appropriate normalized Bellman PDEs, and to study the accuracy of these approximations.

Using the method of *randomized stopping*, it is very useful to rewrite (2.7) in the form of (2.6), by extending A_n and $\mathfrak{A}_n$ as follows. Set

$$\bar{A} = A \times [0,\infty) = \cup_{n=1}^{\infty} \bar{A}_n, \quad \bar{A}_n = A_n \times [0,n],$$

identify $\alpha \in A$ with $(\alpha, 0) \in \bar{A}$, and extend the definition of σ, β, f, g and c by setting

$$\sigma^{\gamma} = \sigma^{\alpha}, \quad \beta^{\gamma} = \beta^{\alpha}, \quad f^{\gamma} = f^{\alpha} + rg, \quad c^{\gamma} = c^{\alpha} + r, \quad \text{for } \gamma = (\alpha, r) \in \bar{A}.$$

Let $\bar{\mathfrak{A}}_n$ denote the set of progressively measurable processes with values in $\bar{\mathfrak{A}}_n$ and set $\bar{\mathfrak{A}} = \cup_n \bar{\mathfrak{A}}_n$. Notice, that if Assumptions (2.1)-(2.2) hold then these assumptions remain valid with $\bar{A}_n$ and $\bar{A}$ in place of A_n and A, with the obvious extension of the metric on A onto $\bar{A}$. Moreover, the following result holds.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Then* $w = \sup_{\gamma \in \bar{\mathfrak{A}}} v^{\gamma}$ *for every* $(s,x) \in [0,T]$*, where* v^{γ} *is defined by* (2.8) *with* $\gamma \in \bar{\mathfrak{A}}$ *in place of* $\alpha \in \mathfrak{A}$.

This theorem, under somewhat stronger assumption is known from [10] when $A = A_n$, $K = K_n$, $m = m_n$ for $n \ge 1$. For the proof we refer to [6].

From [10] one also knows that under some assumptions (more restrictive than Assumptions 2.1-2.2) w satisfies the *normalized Bellman PDE*

$$\sup_{\gamma\in\bar{A}} m^\gamma\left(\frac{\partial}{\partial t}w + L^\gamma w + f^\gamma\right) = 0 \text{ on } H_T \tag{2.10}$$

with terminal condition

$$w(T,x) = g(T,x) \text{ for } x\in\mathbb{R}^d, \tag{2.11}$$

where $m^\gamma = (1+r)^{-1}$ and

$$L^\gamma w = \tfrac{1}{2}\sigma^\gamma_{ip}\sigma^\gamma_{jp}w_{x^ix^j} + \beta^\gamma_i w_{x^i} - c^\gamma w. \tag{2.12}$$

Therefore it is natural to design approximations for w as finite difference approximations for problem (2.10)-(2.11). To this end we fix a constant $K\geq 1$ and make the assumptions below.

Assumption 2.3. There exist a natural number d_1, vectors $\ell_k\in\mathbb{R}^d$ and functions

$$a^\alpha_k : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}_+,\quad b^\alpha_k : \mathbb{R}\times\mathbb{R}^d\to\mathbb{R}_+,\quad \text{for } k=\pm1,\dots,\pm d_1 \text{ and } \alpha\in A,$$

such that $|\ell_k|\leq K$, $\ell_k = -\ell_{-k}$, $a^\alpha_k = a^\alpha_{-k}$, for $k=\pm1,\dots,\pm d_1$, $\alpha\in A$, and

$$\beta^\alpha_i = b^\alpha_k\ell^i_k \tag{2.13}$$

$$\tfrac{1}{2}\sigma^\alpha_{ip}\sigma^\alpha_{jp} = a^\alpha_k\ell^i_k\ell^j_k, \tag{2.14}$$

for $\alpha\in A$ and $i,j=1,2,\dots,d$.

Remark 2.2. For given functions β^α it is easy to find a set of vectors $\{\ell_k\}$ and functions $b^\alpha_k\geq 0$ such that (2.13) holds. We can take, for example, $\ell_{\pm k} = \pm e_k$, with the standard basis $\{e^k\}$ in $\mathbb{R}^d$, and set $b^\alpha_{\pm k} = (\beta^\alpha_k)_\pm$. It is proved in [16] that, if the matrix $\sigma^\alpha\sigma^{\alpha*}$ is uniformly nondegenerate, then there always exist a set of vectors $\ell_k\in\mathbb{R}^d\setminus\{0\}$ and functions a^α_k for $k=\pm1,\dots,\pm d_1$ for some integer d_1 such that $\ell_{-k}=-\ell_k$, $a^\alpha_{-k}=a^\alpha_k\geq 0$ for all k, (2.14) holds, a^α_k are as smooth as $\sigma^\alpha\sigma^{\alpha*}$ is, and $a^\alpha_k\geq\kappa>0$, where κ is a constant. It is also proved in [16] that if all values of the matrix $\sigma^\alpha\sigma^{\alpha*}$ lie in a closed convex polyhedron in the set of nonnegative matrices and the first and second order derivatives in $x\in\mathbb{R}^d$ of $\sigma^\alpha\sigma^{\alpha*}$ are bounded functions, then again there exist $\{\ell_k\}$ and a^α_k satisfying the above assumption such that $\sqrt{a^\alpha_k}$ are Lipschitz continuous in x.

Clearly, (2.13) and (2.14) imply

$$\tfrac{1}{2}\sigma^\alpha_{ip}\sigma^\alpha_{jp}u_{x^ix^j} = a^\alpha_k D^2_{\ell_k}u,\quad \beta^\alpha_i u_{x^i} = b^\alpha_k D_{\ell_k}u$$

for smooth functions u, where we use the notation

$$\mathrm{D}_\ell\, u = u_{x^i}\ell^i \quad \text{for } \ell\in\mathbb{R}^d.$$

Thus setting $a^\gamma_k = a^\alpha_k$ and $b^\gamma_k = b^\alpha_k$ for $\gamma=(\alpha,r)\in\bar{A}$, for the operator L^γ given by (2.12) we have

$$L^\gamma u = a^\gamma_k\,\mathrm{D}^2_{\ell_k}\,u + b^\gamma_k\,\mathrm{D}_{\ell_k}\,u - c^\gamma u,\quad \text{for } \gamma\in\bar{A}.$$

For $\tau > 0$, $h > 0$ and $l \in \mathbb{R}^d$ define

$$\begin{aligned} \delta_\tau\, u(t,x) &:= \tfrac{u(t+\tau,x)-u(t,x)}{\tau}, \quad \tau_T(t) = \tau \wedge (T-t) \\ \delta_\tau^T\, u(t,x) &:= \tfrac{u(t+\tau_T(t),x)-u(t,x)}{\tau}, \\ \delta_{h,l} u(t,x) &:= \tfrac{u(t,x+hl)-u(t,x)}{h}, \\ \Delta_{h,l} u &:= -\delta_{h,l}\delta_{h,-l} u = \tfrac{1}{h}(\delta_{h,l} u + \delta_{h,-l} u). \end{aligned} \tag{2.15}$$

for $t \in [0,T)$, $x \in \mathbb{R}^d$, and consider the finite difference scheme

$$\sup_{\gamma \in \bar{A}} m^\gamma (\delta_\tau^T\, u + L_h^\gamma u + f^\gamma) = 0 \quad \text{on } H_T \tag{2.16}$$

$$u(T,x) = g(x) \quad \text{for } x \in \mathbb{R}^d, \tag{2.17}$$

where

$$L_h^\gamma u = a_k^\gamma\, \Delta_{h,\ell_k}\, u + b_k^\gamma\, \delta_{h,\ell_k}\, u - c^\gamma u.$$

Remark 2.3. Equation (2.10) is often written in the form

$$\max\big[\tfrac{\partial}{\partial t} w + \sup_{\alpha \in A}(L^\alpha w + f^\alpha), g - w\big] = 0 \quad \text{on } H_T, \tag{2.18}$$

and similarly, equation (2.16) can be written as

$$\max\big[\,\delta_\tau^T\, u + \sup_{\alpha \in A}(L_h^\alpha u + f^\alpha), g - u\big] = 0 \quad \text{on } H_T. \tag{2.19}$$

Clearly, equation (2.18) is equivalent to

$$\tfrac{\partial}{\partial t} w + \sup_{\alpha \in A}\,[L^\alpha w + f^\alpha] \le 0, \quad g - w \le 0 \quad \text{on } H_T,$$

$$\tfrac{\partial}{\partial t} w + \sup_{\alpha \in A}\,[L^\alpha w + f^\alpha] = 0, \quad \text{on } \{(t,x) \in H_T : g(t,x) < w(t,x)\},$$

and similarly equation (2.19) is equivalent to

$$\delta_\tau^T u + \sup_{\alpha \in A}\,[\mathrm{L}_h^\alpha\, u + f^\alpha] \le 0, \quad g - u \le 0 \quad \text{on } H_T,$$

$$\delta_\tau^T u + \sup_{\alpha \in A}\,[\mathrm{L}_h^\alpha\, u + f^\alpha] = 0 \quad \text{on } \{(t,x) \in H_T : g(t,x) < u(t,x)\}.$$

Proof. By setting $\varepsilon = \frac{1}{1+r}$ equations (2.10) and (2.16) can be rewritten as

$$\sup_{\varepsilon \in [0,1]} \Big[\varepsilon \sup_{\alpha \in A} \big(\tfrac{\partial}{\partial t} w + L^\alpha w + f^\alpha\big) + (1-\varepsilon)(g - w)\Big] = 0 \quad \text{on } H_T$$

and

$$\sup_{\varepsilon \in [0,1]} \Big[\varepsilon \sup_{\alpha \in A} \big(\delta_\tau^T u + L_h^\alpha u + f^\alpha\big) + (1-\varepsilon)(g - u)\Big] = 0 \quad \text{on } H_T,$$

respectively. Hence we finish the proof of the remark by noticing that for any numbers $p, q \in \mathbb{R}$

$$\sup_{\varepsilon \in [0,1]} (\varepsilon p + (1-\varepsilon) q) = \max(p,q).$$

$\square$

Instead of Assumptions 2.1 and 2.2 we make now the following assumption.

Assumption 2.4. The functions σ^α, a^α_k, b^α_k, f^α and $c^\alpha \geq 0$ are Borel measurable in t and are continuous in $\alpha \in A$ for each $k = \pm 1, \dots, d_1$. Moreover, for $\Psi := \sigma^\alpha, \sqrt{a^\alpha_k}, b^\alpha_k, c^\alpha, f^\alpha, g$ for $\alpha \in A$ and $k = \pm 1, \dots, \pm d_1$ we have

$$|\Psi(t,x) - \Psi(t,y)| \leq K|x-y|, \quad |\psi(t,x)| \leq K \tag{2.20}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Notice that Assumption 2.3 and 2.4 imply Assumptions 2.1 and 2.2. Finally we make an assumptions on Hölder continuity of $\sqrt{a^\alpha_k}$, b^α_k, c^α and f^α.

Assumption 2.5. For $\Psi := \sqrt{a^\alpha_k}, b^\alpha_k, c^\alpha, f^\alpha, g$ for $\alpha \in A$, $k = \pm 1, \dots, \pm d_1$ we have

$$|\psi(t,x) - \psi(t,x)| \leq K|t-s|^{1/2}$$

for all $x \in \mathbb{R}^d$ and $s, t \in \mathbb{R}$.

The following result is the main theorem of the paper. It extends Theorem 2.3 from [15] to the reward function w defined by (2.7).

Theorem 2.4. *Let Assumptions 2.3 through 2.5 hold. Then* (2.16)-(2.17) *has a unique bounded solution $w_{\tau,h}$, and there is a constant N depending only on K, d, d_1, T such that for $\tau, h \leq 1$*

$$|w - w_{\tau,h}| \leq N(\tau^{1/4} + h^{1/2}) \tag{2.21}$$

on $\bar{H}_T$. Moreover, there is a constant λ_0 depending only on K and d_1 such that if $\lambda \geq \lambda_0$ then N is independent of T.

3. On finite difference schemes

Let A be a set and consider for $\alpha \in A$ the finite difference operator

$$\mathrm{L}^\alpha_h = a^\alpha_k \,\Delta_{h,\ell_k} + b^\alpha_k \,\delta_{h,\ell_k} - c^\alpha,$$

where a^α_k, b^α_k, c^α, f^α and g are some functions on $H_\infty := [0,\infty) \times \mathbb{R}^d$ for each $\alpha \in A$ and $k = \pm 1, \dots, \pm d_1$. Recall that $\{\ell_k : k = \pm 1, \pm 2, \dots, \pm d_1\}$ are given vectors in $\mathbb{R}^d$ such that $|\ell_k| \leq K$ for all $k = \pm 1, \dots, \pm d_1$ and $l_k = -l_k$, where $K \geq 1$ is a fixed constant.

Let m^α be a function of $\alpha \in A$ taking values in $(0,1]$. Recall that $H_T = [0,T) \times \mathbb{R}^d$ for a fixed $T \in [0,\infty)$. For fixed $\tau > 0$ and $h > 0$ we are interested in the problem

$$\sup_{\alpha \in A} m^\alpha \left(\delta^T_\tau \, v + \mathrm{L}^\alpha_h \, v + f^\alpha \right) = 0 \quad \text{on} \quad H_T, \tag{3.1}$$

$$v(T,x) = g(T,x) \quad x \in \mathbb{R}^d \tag{3.2}$$

for a function $v = v_{\tau,h}$ defined on $\bar{H}_T = [0,T] \times \mathbb{R}^d$. Notice that problem (3.1)-(3.2) is a collection of separate problems given on each grid

$$\{((t_0 + j\tau) \wedge T, x_0 + (\pm i_1 \ell_1 \pm \dots \pm i_{d_1} \ell_{d_1})h\} \tag{3.3}$$

associated with points $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$, where i_1,...,i_{d_1} and j run through the nonnegative integers. The grid associated with the point $(t_0, x_0) := (0,0)$ is

$$\bar{\mathcal{M}}_{\mathcal{T}} = \{(j\tau \wedge T, \pm i_1 h\ell_1 \pm \dots \pm i_{d_1} h\ell_{d_1}) : j, i_1, \dots, i_{d_1} = 0, 1, \dots\}.$$

Clearly, results obtained for equations on subsets of

$$\mathcal{M}_T := \bar{\mathcal{M}}_T \cap ([0,T) \times \mathbb{R}^d)$$

can be translated into results for for equations on subsets of all other grids of the type (3.3).

In this section we consider the finite difference problems

$$\sup_{\alpha \in A} m^\alpha \left(\delta_\tau^T u + \mathrm{L}_h^\alpha u + f^\alpha \right) = 0 \quad \text{on} \quad Q, \tag{3.4}$$

$$u = g \text{ on } \bar{\mathcal{M}}_T \setminus Q \tag{3.5}$$

and

$$\max \left[\sup_{\alpha \in A} m^\alpha (\delta_\tau^T w + \mathrm{L}_h^\alpha w + f^\alpha), g - w \right] = 0 \quad \text{on} \quad Q, \tag{3.6}$$

$$w = g \text{ on } \bar{\mathcal{M}}_T \setminus Q \tag{3.7}$$

where Q is a fixed subset of $\mathcal{M}_T$ and g is a bounded function on H_∞. Let $\lambda \geq 0$ be a constant and make the following assumptions.

Assumption 3.1. We have $m^\alpha \in (0,1]$, $a_k^\alpha \geq 0$, $b_k^\alpha \geq 0$, $a_k^\alpha = a_{-k}^\alpha$ and $c^\alpha \geq \lambda$ for all $\alpha \in A$, $(t,x) \in H_\infty$ and $k = \pm 1, \pm 2, \dots, \pm d_1$.

Assumption 3.2. For all $k = \pm 1, \dots, \pm d_1$, $\alpha \in A$, $(t,x) \in H_\infty$

$$|m^\alpha a_k^\alpha| + |m^\alpha b_k^\alpha| + |m^\alpha c^\alpha| + |m^\alpha f^\alpha| \leq K.$$

Assumption 3.3. There exists a constant $\rho > 0$ such that

$$m^\alpha (1 + c^\alpha - \lambda) \geq \rho \tag{3.8}$$

on H_∞ for all $\alpha \in A$.

Remark 3.1. Consider $\bar{A} = A \times [0, \infty)$, identify every $\alpha \in A$ with $(\alpha, 0) \in \bar{A}$, and set for $\gamma = (\alpha, r) \in \bar{A}$

$$m^\gamma = m^\alpha (1+r)^{-1}, \quad a_k^\gamma = a_k^\alpha, \quad b_k^\gamma = b_k^\alpha,$$

$$c^\gamma = c^\alpha + \tfrac{r}{m^\alpha}, \quad f^\gamma = f^\alpha + \tfrac{r}{m^\alpha} g.$$

$$L_h^\gamma = a_k^\gamma \, \Delta_{h,\ell_k} + b_k^\gamma \, \delta_{h,\ell_k} - c^\gamma.$$

Then, as Remark 2.3 is shown, it is easy to see that equation (3.6) can be cast into equation (3.4) with $\bar{A}$ in place of A. Clearly, if Assumption 3.1 holds, then it holds also with $\bar{A}$ in place of A. If Assumption 3.3 holds, then it is easy to show that it holds with $\bar{A}$ in place of A and with $\min(\rho, 1)$ in place of ρ. If Assumption 3.2 holds and $|g| \leq K$ on H_∞ then it is easy to see that Assumption 3.2 holds also with $\bar{A}$ in place of A, with constant $2K+1$ in place of K. Thus we obtain the results of this section immediately for both equations (3.4) and (3.6), by proving them only for (3.4) and verifying that the conditions formulated with A hold also with $\bar{A}$ in place of A.

The following simple examples show that if condition (3.8) does not hold then problem (3.4)-(3.5) may have many solutions or may have no solution.

Example 3.2. Let $A = [0, \infty)$, $m^\alpha = (1+\alpha)^{-1}$. Consider the problem

$$\sup_{\alpha \in A} m^\alpha \left(\delta_\tau^T u \right) = 0 \text{ on } \mathcal{M}_T, \quad u = 1 \quad \text{on } \bar{\mathcal{M}}_T \setminus \mathcal{M}_T.$$

Notice that here $\inf_{\alpha \in A} m^\alpha (1 + c^\alpha) = 0$, i.e. the condition (3.8) is violated. If $u : \mathcal{M}_T \to \mathbb{R}$ is any non-increasing function in t, then $m^\alpha \delta_\tau^T u \le 0$. Hence, letting $\alpha \to \infty$, we see that u satisfies the equation. Consequently the solution to the above problem is not unique.

Example 3.3. Let $A = [0, \infty)$, $m^\alpha = (1+\alpha)^{-1}$ and $f^\alpha = 1 + \alpha$. Consider now the equation

$$\sup_{\alpha \in A} m^\alpha (\delta_\tau^T u + f^\alpha) = \sup_{\alpha \in A} m^\alpha \delta_\tau^T u + 1 = 0 \text{ on } \mathcal{M}_T.$$

If u is a solution then we have $m^\alpha \delta_\tau^T u \le 0$. Hence $\sup_{\alpha \in A} m^\alpha \delta_\tau^T u = 0$, which contradicts the equation. Thus the above equation has no solution.

Theorem 3.4. *Let Assumptions 3.1 through 3.3 hold. Let g be a bounded function on $\bar{\mathcal{M}}_T$. Then the finite difference problems* (3.4)-(3.5) *and* (3.6)-(3.7) *admit a unique bounded solution u and w, respectively.*

Proof. By virtue of Remark 3.1 it suffices to prove the lemma for (3.4)-(3.5). Let $\gamma = (0, 1)$ and define ξ recursively as follows: $\xi(T) = 1$, $\xi(t) = \gamma^{-1} \xi(t + \tau_T(t))$ for $t < T$. Then for any function v

$$\delta_\tau^T (\xi v) = \gamma \xi \, \delta_\tau^T v - \nu \xi v, \text{ where } \nu = \tfrac{1-\gamma}{\tau}.$$

Solving (3.4)-(3.5) for u is equivalent to solving

$$v = H[v] := H[(f^\alpha), g, v] := \mathbf{1}_{\bar{\mathcal{M}}_T \setminus Q} \tfrac{1}{\xi} g + \mathbf{1}_Q G[v], \tag{3.9}$$

with $u = \xi v$, where for $\varepsilon > 0$,

$$G[v] := v + \varepsilon \xi^{-1} \sup_\alpha m^\alpha \left(\delta_\tau^T u + \mathrm{L}_h^\alpha u + f^\alpha \right). \tag{3.10}$$

Then

$$G[v] = \sup_\alpha \left[p_\tau^\alpha \, \mathrm{T}_\tau v + p_k^\alpha \, \mathrm{T}_{h, l_k} v + p^\alpha v + \varepsilon m^\alpha \xi^{-1} f^\alpha \right], \tag{3.11}$$

with

$$p_\tau^\alpha = \varepsilon \gamma \tau^{-1} m^\alpha \ge 0, \quad p_k^\alpha = \varepsilon (2h^{-2} a_k^\alpha + h^{-1} b_k^\alpha) m^\alpha \ge 0,$$
$$p^\alpha = 1 - p_\tau^\alpha - \sum_k p_k^\alpha - \varepsilon \nu m^\alpha - \varepsilon m^\alpha c^\alpha.$$

Notice that $p_k^\alpha \le \varepsilon K (h^{-2} + h^{-1})$ and

$$\varepsilon \nu m^\alpha + \varepsilon m^\alpha c^\alpha \le \varepsilon \tau^{-1} + \varepsilon K, \quad p_\tau^\alpha \le \varepsilon \tau^{-1},$$

so for all ε smaller than some ε_0 we have $p^\alpha \ge 0$. Also by taking into account (3.8) we have

$$0 \le \sum_k p_k^\alpha + p^\alpha + p_\tau^\alpha = 1 - \varepsilon m^\alpha (\nu + c^\alpha) \le 1 - \varepsilon (1 \wedge \nu) m^\alpha (1 + c^\alpha)$$
$$\le 1 - \varepsilon (1 \wedge \nu) \rho =: \delta < 1,$$

for sufficiently small $\varepsilon > 0$. Notice also $|m^\alpha f^\alpha| \le K$. Hence H maps bounded functions on $\bar{\mathcal{M}}_T$ into bounded functions on $\bar{\mathcal{M}}_T$. Furthermore

$$|H[v](t,x) - H[w](t,x)| \le \delta \sup_{\bar{\mathcal{M}}_T} |v - w|.$$

Thus the operator H is a contraction on the space of bounded functions on $\bar{\mathcal{M}}_T$. By Banach's fixed point theorem (3.9) has a unique bounded solution. $\square$

Set $\bar{\mathcal{M}}_{T,R} = \{(t,x) \in \bar{\mathcal{M}}_T, |x| \le R\}$ and $\bar{\mathcal{M}}^c_{T,R} = \{(t,x) \in \bar{\mathcal{M}}_T, |x| > R\}$ for $R > 0$.

Remark 3.5. Let v be a function on $\bar{\mathcal{M}}_T$. The operator H defined by (3.9) has the following property: if there exists $R > 0$ such that $v = f^\alpha = 0$ on $\bar{\mathcal{M}}^c_{T,R}$ for all $\alpha \in A$, then there exists R' such that

$$H[(f^\alpha), g, v](t,x) = 0 \quad \text{on } \bar{\mathcal{M}}^c_{T,R'}.$$

Corollary 3.6. *Let Assumptions 3.1 through 3.3 hold. Let u be the bounded solution of (3.4)-(3.5) with $Q = \mathcal{M}_T$. Assume there exists $R > 0$ such that for all $\alpha \in A$*

$$f^\alpha = g = 0 \quad \textit{on } \bar{\mathcal{M}}^c_{T,R}.$$

Then

$$\lim_{r\to\infty} \sup_{\bar{\mathcal{M}}^c_{T,r}} |u(t,x)| = 0.$$

Proof. Let ξ be defined as in the proof of Theorem 3.4 and let $v = \xi u$. For a fixed (f^α) and g we define $H^n[v]$ for functions v on $\bar{\mathcal{M}}_T$ recursively in n as follows: $H^1[v] = H^1[(f^\alpha), g, v]$ and $H^n[v] = H^1[H^{n-1}[v]]$ for $n \ge 2$. From the proof of Theorem 3.4 we see that H is a contraction on the space of bounded functions on $\bar{\mathcal{M}}_T$. Hence for any $\varepsilon > 0$ there is n_0 such that

$$\sup_{\bar{\mathcal{M}}_T} |H^n[0] - v| < \varepsilon, \text{ for } n \ge n_0.$$

By Remark 3.5 there exist R_ε such that $H^{n_0}[0] = 0$ on $\bar{\mathcal{M}}^c_{T,R_\varepsilon}$. Hence

$$\sup_{\bar{\mathcal{M}}^c_{T,R_\varepsilon}} |v| < \varepsilon,$$

which proves the corollary. $\square$

For the next lemma we need some remarks from [14]. Let D^n_x denote the collection of all n-th order derivatives in x.

Remark 3.7. For any sufficiently smooth function $\eta(x)$ by Taylor's formula

$$|\,\mathrm{L}^\alpha \eta(x) - \mathrm{L}^\alpha_h \eta(x)| \le N(h^2 \sup_{B_K(x)} |\,\mathrm{D}^4_x \eta| + h \sup_{B_K(x)} |\,\mathrm{D}^2_x \eta|),$$

where $B_K(x)$ is the ball of radius K centered at x.

Remark 3.8. Let us introduce T' as the least integer multiple of τ not less than T. Notice that problem (3.4)-(3.5) can be rewritten as

$$\sup_{\alpha\in A} (\delta_\tau \tilde{u} + \mathrm{L}^\alpha_h \tilde{u} + f^\alpha) = 0, \quad \text{on } Q$$

$$\tilde{u} = \tilde{g} \quad \text{on } \bar{\mathcal{M}}_{T'} \setminus Q,$$

where $\tilde{u}(t,x) = u(t,x)$ on $\mathcal{M}_{T'}$, $\tilde{u}(T',x) = u(T,x)$, $\tilde{g} = g$ on $\mathcal{M}_{T'}$ and $\tilde{g}(T',x) = g(T,x)$. Observe that

$$\delta_\tau \tilde{u} = \delta_\tau^{T'} \tilde{u} = \delta_\tau^T u \text{ on } \mathcal{M}_{T'}.$$

Lemma 3.9. *Assume that a_k^α, b_k^α and c^α satisfy Assumptions 3.1 and 3.2. Let f_1^α and f_2^α be functions on $A \times \mathcal{M}_T$ such that*

$$\sup_\alpha m^\alpha f_2^\alpha < \infty, \quad f_1^\alpha \le f_2^\alpha \quad \text{on } Q \text{ for every } \alpha \in A.$$

Let u_1 and u_2 be functions on $\bar{\mathcal{M}}_T$ such that for some constants $\mu \ge 0$ and $C \ge 0$ the functions $u_1(t,x)e^{-\mu|x|}$ and $u_2(t,x)e^{-\mu|x|}$ are bounded on $\bar{\mathcal{M}}_T$ and

$$\begin{aligned} &\sup_\alpha m^\alpha \left(\delta_\tau^T u_1 + \mathrm{L}_h^\alpha u_1 + f_1^\alpha + C\right) \\ &\ge \sup_\alpha m^\alpha \left(\delta_\tau^T u_2 + \mathrm{L}_h^\alpha u_2 + f_2^\alpha\right) \text{ on } Q, \end{aligned} \tag{3.12}$$

$$u_1 \le u_2 \quad \text{on } \bar{\mathcal{M}}_T \setminus Q. \tag{3.13}$$

Assume also that $h \le 1$. Then there exists a constant τ^* depending only on K, d_1, μ such for $\tau \in (0, \tau^*)$

$$u_1 \le u_2 + T'C \text{ on } \bar{\mathcal{M}}_T. \tag{3.14}$$

If u_1, u_2 are bounded on Q then (3.14) holds for all positive τ and h.

Proof. By using Remark 3.8 we may assume that $T = T'$ and $\delta_\tau^T = \delta_\tau$. Let $w = u_1 - u_2 - C(T' - t)$. Then from (3.12)

$$\sup_\alpha m^\alpha (\delta_\tau w + \mathrm{L}_h^\alpha w) \ge 0, \text{ on } Q.$$

Notice that, as in (3.11) with $\gamma = 1$ (hence $\xi = 1$ and $\nu = 0$) and $f^\alpha = 0$, we have

$$G[w] = w + \varepsilon \sup_\alpha m^\alpha \left(\delta_\tau^T w + \mathrm{L}_h^\alpha w\right) = \sup_{\alpha \in A} [p_\tau^\alpha \mathrm{T}_\tau w + p_k^\alpha \mathrm{T}_{h,l_k} w + p^\alpha w],$$

with

$$p_\tau^\alpha = \varepsilon \tau^{-1} m^\alpha \ge 0, \quad p_k^\alpha = \varepsilon m^\alpha (2h^{-2} a_h^\alpha + h^{-1} b_k^\alpha) \ge 0$$

and

$$p^\alpha = 1 - p_\tau^\alpha - \sum_k p_k^\alpha - \varepsilon m^\alpha c^\alpha,$$

where one can see that also $p^\alpha \ge 0$ if ε is sufficiently small. Thus G is a monotone operator in the sense that for any $\psi \ge w$ on $\bar{\mathcal{M}}_T$ we have $G[\psi] \ge G[w]$ on $\mathcal{M}_T$. So for any sufficiently small fixed $\varepsilon > 0$ and $\psi \ge w$ on $\bar{\mathcal{M}}_T$

$$\psi + \varepsilon \sup_\alpha m^\alpha (\delta_\tau \psi + \mathrm{L}_h^\alpha \psi) \ge w, \text{ on } Q. \tag{3.15}$$

Let $\gamma \in (0,1)$. Use ξ from the proof of Theorem 3.4. Then

$$\delta_\tau^T \xi = \xi \tfrac{1}{\tau}(\gamma - 1).$$

Let $\eta(x) = \cosh(\mu|x|)$ and $\zeta = \eta\xi$. Introduce

$$N_0 = \sup_{\bar{\mathcal{M}}_T} \tfrac{w_+}{\zeta}.$$

Due to the assumption that $u_1(t,x)e^{-\mu|x|}$ and $u_2(t,x)e^{-\mu|x|}$ are bounded on $\bar{\mathcal{M}}_T$, we have $N_0 < \infty$. Our aim now is to show that, in fact $N_0 = 0$.

By Remark 3.7, taking into account that for every $\mu > 0$ and integer $n \geq 1$ there is a constant N such that for all $x \in \mathbb{R}^d$

$$|D_x^n \cosh(\mu|x|)| \leq N \cosh(\mu|x|),$$

we get

$$\begin{aligned} m^\alpha \,\mathrm{L}_h^\alpha\, \eta(x) &\leq m^\alpha L^\alpha \eta(x) + N_1(h^2+h)\cosh(\mu|x|+\mu K) \\ &\leq N_2 \cosh(\mu|x|+\mu K) \leq N_3 \cosh(\mu|x|), \end{aligned} \tag{3.16}$$

where N_1 and N_2 are constants depending on μ, d_1, K, and

$$N_3 := N_2 \sup_{x\in\mathbb{R}^d} \tfrac{\cosh(\mu|x|+\mu K)}{\cosh(\mu|x|)} < \infty.$$

Thus

$$m^\alpha\,(\delta_\tau\,\zeta + \mathrm{L}_h^\alpha\,\zeta) \leq \zeta[\tau^{-1}(\gamma-1)+N_3],$$

Let

$$\psi := N_0\zeta \geq \zeta \tfrac{w_+}{\zeta} \geq w.$$

Then by (3.15)

$$w \leq \psi + \varepsilon \sup_\alpha m^\alpha(\delta_\tau^T\,\psi + \mathrm{L}_h^\alpha\,\psi) \leq \zeta(N_0 + \varepsilon\kappa) \tag{3.17}$$

holds on Q, where $\kappa = \kappa(\gamma) = \tau^{-1}(\gamma-1)+N_3$. Notice that $\kappa(0) < 0$ for $\tau < \tau^* := N_3^{-1}$, and $\kappa(1) > 0$. So there is a $\gamma \in (0,1)$, which we choose now, such that $\kappa < 0$ and $N_0 + \varepsilon\kappa > 0$. Thus by (3.17) and (3.13)

$$w \leq \zeta(N_0+\varepsilon\kappa) \text{ on } \bar{\mathcal{M}}_T.$$

Hence

$$N_0 = \sup_{\bar{\mathcal{M}}_T} \tfrac{w_+}{\zeta} \leq N_0 + \varepsilon\kappa, \tag{3.18}$$

which implies $N_0 = 0$, since $\varepsilon\kappa < 0$. This completes the proof of the first assertion of the lemma.

Assume now that u_1 and u_2 are bounded on Q. Then we can take $\mu = 0$, i.e., $\eta = 1$. We do not need estimate (3.16), hence there is no restriction on h. We can take $N_3 = 0$ and hence we do not need any restriction on τ. □

Corollary 3.10. *Let Assumptions 3.1 through 3.3 hold. Let Q be a subset of $\bar{\mathcal{M}}_T$. Assume that g is a bounded function on $\bar{\mathcal{M}}_T$ and let u and w denote the unique bounded solutions of* (3.4)-(3.5) *and* (3.6)-(3.7)*, respectively. Let ψ be a function on $\bar{\mathcal{M}}_T$ such that for some constant $\mu \geq 0$ the function $e^{-\mu x}\psi(t,x)$ is bounded on $\bar{\mathcal{M}}_T$. Then the following statements hold:*

(i) *Let*

$$\delta_\tau^T\,\psi + \mathrm{L}_h^\alpha\,\psi + f^\alpha \leq 0 \quad \text{on } Q \text{ for each } \alpha \in A.$$

Then $\psi \geq g$ on $\bar{\mathcal{M}}_T \setminus Q$ implies $\psi \geq u$ on $\bar{\mathcal{M}}_T$, and $\psi \geq g$ on $\bar{\mathcal{M}}_T$ implies $\psi \geq w$ on $\bar{\mathcal{M}}_T$.

(ii) *Let*

$$\delta_\tau^T\,\psi + \mathrm{L}_h^\alpha\,\psi + f^\alpha \geq 0 \quad \text{on } Q \text{ for some } \alpha \in A,$$

and $\psi \geq g$ on $\bar{\mathcal{M}}_T \setminus Q$. Then $\psi \leq u$ and $\psi \leq w$ on $\bar{\mathcal{M}}_T$.

Proof. The statements concerning u follow immediately from the previous lemma. Hence the statements concerning w follow by Remark 3.1. □

Let us consider now problem (3.1)-(3.2) and

$$\max\left[\sup_{\alpha\in A} m^{\alpha}(\delta^T_{\tau}\, w + \mathrm{L}^{\alpha}_h\, w + f^{\alpha}), g - w\right] = 0 \quad \text{on} \quad H_T, \tag{3.19}$$

$$w(T,x) = g(T,x) \quad \text{for } x \in \mathbb{R}^d. \tag{3.20}$$

Corollary 3.11. *Let Assumptions 3.1 through 3.3 hold. Let $c_1 \geq 0$ be a constant such that*

$$\tau^{-1}(e^{c_1\tau} - 1) \leq \lambda. \tag{3.21}$$

Then problem (3.1)-(3.2) *has a unique bounded solution u, and*

$$|u(t,x)| \leq N^* + e^{-c_1(T-t)} \sup_{x\in\mathbb{R}^d} |g(T,x)|, \tag{3.22}$$

holds on $\bar{H}_T$, where

$$N^* = \begin{cases} K\rho^{-1}(\lambda^{-1}(1-e^{-\lambda T'}) + 1) & \text{when} \quad \lambda > 0, \\ K\rho^{-1}(T'+1) & \text{when} \quad \lambda = 0. \end{cases} \tag{3.23}$$

In addition to the above conditions assume that $|g| \leq K$ on H_T. Then problem (3.19)-(3.20) *has a unique bounded solution w and* (3.22) *holds for w in place of u, with $2K+1$ in place of K in* (3.23).

Proof. It suffices to prove the corollary for problem (3.1)-(3.2). Hence we get the statement of the corollary also for (3.19)-(3.20), by rewriting it into the form of (3.1)-(3.2), as it is explained in Remark 3.1. By Theorem 3.4 problem (3.1)-(3.2) has a unique solution u, which is bounded on each grid defined by (3.3). Hence it suffices to prove estimate (3.22) on the grid $\bar{\mathcal{M}}_T$. As before, by virtue of Remark 3.8 we may assume that $T = T'$ and so $\delta^T_\tau = \delta_\tau$. Let $\lambda > 0$. Then set $N := \sup_x |g(T,x)|$ and

$$\xi(t) = K\rho^{-1}\{\lambda^{-1}(1-e^{-\lambda(T-t)}) + 1\} + e^{-c_1(T-t)}N.$$

Then on $\mathcal{M}_T$

$$\begin{aligned} m^{\alpha}(\delta_\tau\, \xi + \mathrm{L}^{\alpha}_h\, \xi + f^{\alpha}) &= m^{\alpha}\{\delta_\tau\, \xi - \lambda\xi - (c^{\alpha}-\lambda)\xi + f^{\alpha}\} \\ = -K\rho^{-1}m^{\alpha}&\left[e^{\lambda t-\lambda T}\big(\frac{e^{\lambda\tau}-1}{\lambda\tau} - 1\big) + 1\right] - m^{\alpha}(c^{\alpha}-\lambda)\xi \\ +m^{\alpha}N\tau^{-1}(e^{c_1\tau}-1)&e^{c_1(T-t)} - m^{\alpha}\lambda N e^{-c_1(T-t)} + m^{\alpha}f^{\alpha}. \end{aligned}$$

Thus, due to

$$\frac{e^{\lambda\tau}-1}{\lambda\tau} > 1, \quad \xi \geq K\rho^{-1}, \quad m^{\alpha}f^{\alpha} \leq K$$

and conditions (3.8) and (3.21) we have

$$m^{\alpha}(\delta_\tau\, \xi + \mathrm{L}^{\alpha}_h\, \xi + f^{\alpha}) \leq -K\rho^{-1}m^{\alpha}(1 + c^{\alpha} - \lambda) + K \leq 0 \quad \text{on } \mathcal{M}_T.$$

Clearly

$$\xi(T) \geq \sup_x |g(T,x)| \geq g(T,x).$$

Hence applying Lemma 3.9 with u and ξ in place of u_1 and u_2, respectively, we get $u \leq \xi$ on $\bar{\mathcal{M}}_T$. Similarly, by using $-\xi$ in place of ξ, we get $u \geq -\xi$ on $\bar{\mathcal{M}}_T$. If $\lambda = 0$ then $c_1 = 0$, and taking $\xi = K\rho^{-1}(T+1) + N$ we get (3.22) in the same way as above. □

Finally we can show that Lemma 3.8 of [14] remains valid in our setting.

Lemma 3.12. *Assume that Assumptions 3.1 through 3.3 hold. Let u be the solution of* (3.4)-(3.5) *for a bounded function g on $\mathbb{R}^d$. For every integer $n \geq 1$ let f_n^α and g_n be functions on $A \times H_T$ and on $\mathbb{R}^d$, respectively such that*

$$\sup_{\alpha\in A}\sup_{\bar H_T}|m^\alpha f_n^\alpha| + \sup_{\mathbb{R}^d}|g_n| \leq K \quad \text{for all } n \geq 1,$$

$$\lim_{n\to\infty}(\sup_\alpha m^\alpha|f^\alpha - f_n^\alpha| + |g-g_n|) = 0 \quad \text{for every } t\in[0,T],\ x\in\mathbb{R}^d.$$

Then $u_n \to u$ on $\bar{\mathcal{M}}_T$ as $n\to\infty$, where u_n is the bounded solution of (3.4)-(3.5) *with f_n^α and g_n in place of f^α and g, respectively.*

Proof. Having Theorem 3.4 and Corollary 3.11 at our disposal we can get this lemma in the same way as Lemma 3.8 in [14] is proved: Since by Corollary 3.11 u_n is bounded uniformly in n, any subsequence of $\{u_n\}$ contains a subsequence converging to a solution of (3.4)-(3.5), which is unique and equals u. Therefore the whole sequence u_n converges to u. □

4. Gradient Estimates for Finite Difference Schemes

Thorough this section we assume that Assumption 3.1 holds. Recall that T' denotes the smallest integer multiple of τ which is greater than or equal to T. For a fixed number $\varepsilon \in (0, Kh]$ and a unit vector $l \in R^d$, set $h_r = h$ for $r = \pm1, \dots, \pm d_1$ and $h_r = \varepsilon$ for $r = \pm(d_1+1)$, and $\ell_{\pm(d_1+1)} = \pm l$. Define

$$\bar{\mathcal{M}}_T(\varepsilon) := \{(t, x+i\varepsilon l) : (t,x)\in\bar{\mathcal{M}}_T, i = 0, \pm1, \dots\},$$
$$\mathcal{M}_T(\varepsilon) := \bar{\mathcal{M}}_T(\varepsilon) \cap ([0,T)\times\mathbb{R}^d).$$

Let $Q \subset \bar{\mathcal{M}}_T(\varepsilon)$ be a nonempty finite set. Define $Q' = Q \cap ([0,T)\times\mathbb{R}^d)$,

$$Q_\varepsilon^0 = \{(t,x) : (t+\tau_T(t), x)\in Q', (t, x+h_r\ell_r)\in Q, \forall r = \pm1,\dots,\pm(d_1+1)\}$$

and $\partial_\varepsilon Q := Q \setminus Q_\varepsilon^0$.

Assumption 4.1. For $r = \pm1, \dots, \pm(d_1+1)$ and $\alpha \in A$ on Q_ε^0 we have

$$|\,\delta_{h_r,\ell_r} b_k^\alpha| \leq K, \quad m^\alpha|\,\delta_{h_r,\ell_r} f^\alpha| \leq K, \quad m^\alpha|\,\delta_{h_r,\ell_r} c^\alpha| \leq K, \tag{4.1}$$

$$|\,\delta_{h_r,\ell_r} a_k^\alpha| \leq K\sqrt{a_k^\alpha} + Kh. \tag{4.2}$$

The following estimate plays a crucial role in the proof of Theorem 2.4. It generalizes Theorem 5.2 from [15].

Theorem 4.1. *Let Assumptions 3.1, 3.3 and 4.1 hold. Let u be a function on $\bar{\mathcal{M}}_T(\varepsilon)$ such that it satisfies (3.4) with Q' in place of Q. Then there is a constant $N^* \geq 0$, depending only on d_1 and K such that for any constant $c_o \geq 0$ satisfying*

$$\lambda + \tfrac{1}{\tau}(1 - e^{-c_o\tau}) \geq N^* + 1, \tag{4.3}$$

we have

$$|\delta_{\varepsilon,\pm l}u| \leq \sqrt{\tfrac{2N^*}{\rho}}e^{c_o(T+\tau)}\big(1 + \max_Q|u| + \max_{r=\pm1,\dots,\pm(d_1+1)}\max_{\partial_\varepsilon Q}|\delta_{h_r,\ell_r}u|\big) \quad \text{on } Q. \tag{4.4}$$

In addition to the above conditions assume that g is a function on $\bar H_T$ such that $|\,\delta_{h_r,l_r} g| \leq K$ on Q_ε^0 for every $r = \pm1,\dots,\pm(d_1+1)$. Let w be a function on $\bar{\mathcal{M}}_T(\varepsilon)$ that satisfies (3.6) *with Q' in place of Q. Then the above statement holds also for w in place of u.*

Proof. We follow the proof of Theorem 5.2 from [15] with some changes. Let

$$v_r = \delta_{h_r,l_r}\, v, \quad v = \xi u, \quad \xi(t) = \left\{ \begin{array}{ll} e^{c_o t} & t < T, \\ e^{c_o T'} & t = T \end{array} \right\},$$

where T' denotes the smallest multiple of τ that is not less than T. Let $(t_0, x_0) \in Q$ be the point where

$$V = \sum_r (v_r^-)^2$$

is maximized. By definition, for any $(t,x) \in Q^o_\varepsilon$ we know that

$$(t, x + h_r \ell_r) \in Q.$$

Clearly, either

$$v_r(t,x) \le 0 \quad \text{or} \quad -v_r(t,x) = v_{-r}(t, x + h_r\ell_r) \le 0.$$

Consequently,

$$|v_r(t,x)| \le V^{1/2}(t_0,x_0).$$

Hence

$$M_1 := \sup_{Q,r} |v_r| \le \sup_{\partial_\varepsilon Q, r} |v_r| + V^{1/2}(t_0,x_0), \tag{4.5}$$

$$|\delta_{\varepsilon,\pm l} u| \le e^{c_0 T'} \sup_{\partial_\varepsilon Q, r} |\delta_{h_r,\ell_r} u| + V^{1/2}(t_0,x_0) \quad \text{on } Q. \tag{4.6}$$

So we need only estimate V on Q. If (t_0, x_0) belongs to $\partial_\varepsilon Q$, then the conclusion of the theorem is clearly true. Thus, we may assume that $(t_0,x_0) \in Q^0_\varepsilon$. For any $\varepsilon_0 > 0$ there exists $\alpha_0 \in A$ such that at (t_0, x_0),

$$m^{\alpha_0}\left(\delta^T_\tau\, u + a^{\alpha_0}_k\, \Delta_{h,\ell_k}\, u + b^{\alpha_0}_k\, \delta_{h,\ell_k}\, u - c^{\alpha_0} u + f^{\alpha_0}\right) + \varepsilon_0 \ge 0,$$

and so for some $\varepsilon \in [0, \varepsilon_0]$

$$m^{\alpha_0}\left(\delta^T_\tau\, u + a^{\alpha_0}_k\, \Delta_{h,\ell_k}\, u + b^{\alpha_0}_k\, \delta_{h,\ell_k}\, u - c^{\alpha_0} u + f^{\alpha_0}\right) + \varepsilon = 0. \tag{4.7}$$

Furthermore (thanks to the fact that $(t_0,x_0) \in Q^0_\varepsilon$)

$$\mathrm{T}_{h_r,\ell_r}\left[m^{\alpha_0}\left(\delta^T_\tau\, u + a^{\alpha_0}_k\, \Delta_{h,\ell_k}\, u + b^{\alpha_0}_k\, \delta_{h,\ell_k}\, u - c^{\alpha_0} u + f^{\alpha_0}\right)\right] \le 0, \tag{4.8}$$

where $\mathrm{T}_{h,l}\,\varphi(t,x) := \varphi(t, x+hl)$ for any number h, vector $l \in \mathbb{R}^d$ and function φ defined at (t,x) and $(t, x+hl)$. Here and below (t_0,x_0) is fixed and for simplicity of notation it is omitted in the arguments of the functions. We subtract (4.7) from (4.8) and divide by h_r to obtain that for each r

$$m^{\alpha_0}\,\delta_{h_r,\ell_r}\left(\delta^T_\tau\, u + a^{\alpha_0}_k\, \Delta_{h,\ell_k}\, u + b^{\alpha_0}_k\, \delta_{h,\ell_k}\, u + f^{\alpha_0} - c^{\alpha_0} u\right) - \tfrac{\varepsilon}{h_r} \le 0.$$

By the discrete Leibnitz rule

$$\begin{aligned} m^{\alpha_0}\left(\delta_\tau(\xi^{-1} v_r) + \xi^{-1}\left[a^{\alpha_0}_k\, \Delta_{h,\ell_k}\, v_r + I_{1r} + I_{2r} + I_{3r}\right] + \delta_{h_r,l_r}\, f^{\alpha_0}\right) \\ -\xi^{-1}\,\delta_{h_r,\ell_r}(m^{\alpha_0} c^{\alpha_0} v) - \tfrac{\varepsilon}{h_r} \le 0, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} I_{1r} &= (\delta_{h_r,\ell_r}\, a^{\alpha_0}_k)\, \Delta_{h,\ell_k}\, v, \\ I_{2r} &= h_r(\delta_{h_r,\ell_r}\, a^{\alpha_0}_k)\, \Delta_{h,\ell_k}\, v_r, \\ I_{3r} &= b^{\alpha_0}_k\, \delta_{h,\ell_k}\, v_r + (\delta_{h_r,\ell_r}\, b^{\alpha_0}_k)\, \mathrm{T}_{h_r,\ell_r}\, \delta_{h,\ell_k}\, v. \end{aligned}$$

Notice that

$$\begin{aligned} 0 &\geq \Delta_{h,\ell_k} \sum_r (v_r^-)^2 = 2v_r^- \Delta_{h,\ell_k} v_r^- + \sum_r \left[(\delta_{h,\ell_k} v_r^-)^2 + (\delta_{h_k,\ell_{-k}} v_r^-)^2\right] \\ &\geq -2v_r^- \Delta_{h,\ell_k} v_r + \sum_r (\delta_{h,\ell_k} v_r^-)^2 + \sum_r (\delta_{h,-\ell_k} v_r^-)^2, \end{aligned}$$

which gives

$$0 \leq v_r^- \Delta_{h,\ell_k} v_r \tag{4.10}$$

and

$$I := \sum_r a_k^{\alpha_0} (\delta_{h,\ell_k} v_r^-)^2 \leq v_r^- a_k^{\alpha_0} \Delta_{h,\ell_k} v_r.$$

Multiplying (4.9) by ξv_r^- and summing up in r we get

$$\begin{aligned} &m^{\alpha_0}\Big(\xi v_r^- \delta_\tau(\xi^{-1} v_r) + \tfrac{1}{2} a_k^{\alpha_0} v_r^- \Delta_{h,\ell_k} v_r + \tfrac{1}{2} I + v_r^- [I_{1r} + I_{2r} \\ &\quad + I_{3r} + \xi\, \delta_{h_r,\ell_r} f^{\alpha_0}]\Big) - v_r^- \delta_{h_r,\ell_r}(m^{\alpha_0} c^{\alpha_0} v) - \xi v_r^- \tfrac{\varepsilon}{h_r} \leq 0. \end{aligned} \tag{4.11}$$

Since $-v_r^- v_r = \sum_r (v_r^-)^2$, $m^{\alpha_0}\, \delta_{h_r,\ell_r} f^{\alpha_0} \geq -K$ and $m^{\alpha_0} |\, \delta_{h_r,\ell_r} c^{\alpha_0}| \leq K$, we have

$$\begin{aligned} &m^{\alpha_0} v_r^- \xi\, \delta_{h_r,\ell_r} f^{\alpha_0} - v_r^- \delta_{h_r,\ell_r}(m^{\alpha_0} c^{\alpha_0} v) \\ &\quad = m^{\alpha_0} v_r^- \xi\, \delta_{h_r,\ell_r} f^{\alpha_0} - m^{\alpha_0} v_r^- (\delta_{h_r,l_r} c^{\alpha_0})\, \mathrm{T}_{h_r,\ell_r} v - m^{\alpha_0} v_r^- c^{\alpha_0} v_r \\ &\quad \geq -e^{c_0 T'} K \sum_r v_r^- - m^{\alpha_0} v_r^- |\, \delta_{h_r,\ell_r} c^{\alpha_0}||\, \mathrm{T}_{h_r,\ell_r} v| + m^{\alpha_0} c^{\alpha_0} \sum_r (v_r^-)^2 \\ &\quad \geq -e^{c_o T'} 2K(d_1+1)M_1 - 2(d_1+1)KM_1M_0 + m^{\alpha_0} c^{\alpha_0} V, \end{aligned}$$

where

$$M_0 := \max_Q |v|.$$

Since V attains its maximum at $(t_0, x_0) \in Q_\varepsilon^0$ we have

$$\begin{aligned} 0 &\geq \sum_r \delta_{h,\ell_k} (v_r^-)^2 = 2v_r^- \delta_{h,\ell_k} v_r^- + \sum_r h_k (\delta_{h,\ell_k} v_r^-)^2 \\ &\geq 2v_r^- \delta_{h,\ell_k} v_r^- \geq -2v_r^- \delta_{h,\ell_k} v_r. \end{aligned}$$

Next recall that $b_k^\alpha \geq 0$ and $|\, \delta_{h_r,l_r} b_k^\alpha| \leq K$. Therefore

$$-v_r^- b_k^{\alpha_0}\, \delta_{h,\ell_k} v_r \leq 0,$$

and

$$v_r^- I_{3r} \geq -v_r^- |\, \delta_{h_r,\ell_r} b_k^{\alpha_0}||\, \mathrm{T}_{h_r,\ell_r} \delta_{h,\ell_k} v| \geq -4Kd_1(d_1+1)M_1^2.$$

By the discrete Leibnitz rule

$$\begin{aligned} \xi v_r^- \delta_\tau^T(\xi^{-1} v_r) &= \xi v_r^- \left[\xi^{-1}(t_0+\tau)\, \delta_\tau^T v_r + v_r\, \delta_\tau^T \xi^{-1}\right] \\ &= e^{-c_o\tau} v_r^- \delta_\tau^T v_r - V\xi\, \delta_\tau^T \xi^{-1} \geq -V\xi\, \delta_\tau^T \xi^{-1} \\ &= \nu V, \end{aligned}$$

where

$$\nu = \nu(c_o) = \tfrac{1}{\tau}(1 - e^{-c_o\tau}).$$

Using the above estimates we get

$$m^{\alpha_0}(\nu+c^{\alpha_0})V+\tfrac{1}{2}m^{\alpha_0}a_k^{\alpha_0}v_r^-\,\Delta_{h,\ell_k}v_r+\tfrac{1}{2}m^{\alpha_0}I$$
$$+v_r^-m^{\alpha_0}[I_{1r}+I_{2r}]-\xi v_r^-\tfrac{\varepsilon}{h_r}\le 2(d_1+1)KM_1(e^{c_oT'}+M_0+2d_1m^{\alpha_0}M_1).$$

Hence

$$m^{\alpha_0}(\nu+c^{\alpha_0})V\le 2(d_1+1)KM_1(e^{c_oT'}+M_0+2d_1m^{\alpha_0}M_1)$$
$$+m^{\alpha_0}v_r^-|I_{1r}|+m^{\alpha_0}v_r^-|I_{2r}|-\tfrac{1}{2}m^{\alpha_0}a_k^{\alpha_0}v_r^-\,\Delta_{h,\ell_k}v_r-\tfrac{1}{2}m^{\alpha_0}I+\xi v_r^-\tfrac{\varepsilon}{h_r}.$$

Define

$$J_1:=v_r^-|(\delta_{h_r,\ell_r}a_k^{\alpha_o})\,\Delta_{h,\ell_k}v|-\tfrac{1}{4}\sum_r a_k^{\alpha_0}(\delta_{h,\ell_k}v_r^-)^2,$$
$$J_2:=J_3-\tfrac{1}{2}a_k^{\alpha_0}v_r^-\,\Delta_{h,\ell_k}v_r-\tfrac{1}{4}\sum_r a_k^{\alpha_0}(\delta_{h,\ell_k}v_r^-)^2,$$
$$J_3:=h_rv_r^-|(\delta_{h_r,\ell_r}a_k^{\alpha_0})\,\Delta_{h,\ell_k}v_r|.$$

Then we can rewrite the above inequality as

$$m^{\alpha_0}(\nu+c^{\alpha_0})V\le 2(d_1+1)KM_1(e^{c_oT'}+M_0+2d_1m^{\alpha_0}M_1)$$
$$+m^{\alpha_0}(J_1+J_2)+\xi v_r^-\tfrac{\varepsilon}{h_r}. \tag{4.12}$$

So we need to estimate J_1, J_2. We turn our attention to J_1. Using condition (4.2), noticing that $h|\,\Delta_{h,\ell_k}v|\le 2M_1$ and

$$|\,\Delta_{h,\ell_k}v|\le\sum_r|\,\delta_{h,\ell_k}v_r^-|+\sum_r|\,\delta_{h_k,\ell_{-k}}v_r^-|,$$

we have

$$\begin{aligned}
v_r^-|(\delta_{h_r,l_r}a_k^{\alpha_0})\,\Delta_{h,\ell_k}v| &\le N_1M_1|(\sqrt{a_k^{\alpha_0}}+h)\,\Delta_{h,\ell_k}v|\\
&\le N_1M_1\sqrt{a_k^{\alpha_0}}|\,\Delta_{h,\ell_k}v|+N_2M_1^2\\
&\le 2N_1M_1\sqrt{a_k^{\alpha_0}}\sum_r|\,\delta_{h,\ell_k}v_r^-|+N_2M_1^2\\
&\le N_3M_1^2+\tfrac{1}{4}\sum_r a_k^{\alpha_0}(\delta_{h,\ell_k}v_r^-)^2,
\end{aligned}$$

where N_1, N_2 and N_3 are constants depending only on d_1 and K. So

$$J_1\le N_3M_1^2. \tag{4.13}$$

Next we estimate J_3. Since $h_r\le Kh$ for all r, by condition (4.2)

$$J_3\le K^2hv_r^-\sqrt{a_k^{\alpha_0}}|\,\Delta_{h,\ell_k}v_r|+K^2h^2\sum_k v_r^-|\,\Delta_{h,\ell_k}v_r|.$$

Hence using $h^2|\,\Delta_{h,\ell_k}v_r|\le 4M_1$ and $|a|=2a^-+a$, we get

$$J_3\le 2K^2hv_r^-\sqrt{a_k^{\alpha_0}}(\Delta_{h,\ell_k}v_r)^-+K^2hv_r^-\sqrt{\alpha_k^{\alpha_0}}\,\Delta_{h,\ell_k}v_r+8K^2d_1M_1^2.$$

Notice that the summations in r above can be restricted to $\{r: v_r<0\}$. For these r we have

$$h(\Delta_{h,\ell_k}v_r)^-\le h|\,\Delta_{h,\ell_k}v_r^-|\le|\,\delta_{h,\ell_k}v_r^-|+|\,\delta_{h_k,\ell_{-k}}v_r^-|.$$

Hence

$$J_3 \le 4K^2 v_r^- \sqrt{a_k^{\alpha_0}}|\delta_{h_k,\ell_k} v_r^-| + K^2 v_r^- h\sqrt{a_k^{\alpha_0}}\,\Delta_{h,\ell_k}\, v_r + 8K^2 d_1 M_1^2$$
$$\le NM_1^2 + \tfrac14 \sum_r a_k^{\alpha_0}(\delta_{h,\ell_k}\, v_r^-)^2 + K^2 v_r^- h\sqrt{a_k^{\alpha_0}}\,\Delta_{h,\ell_k}\, v_r,$$
$$J_2 \le NM_1^2 - \tfrac12\big(a_k^{\alpha_0} - 2K^2 h\sqrt{a_k^{\alpha_0}}\big)v_r^-\,\Delta_{h,\ell_k}\, v_r.$$

By (4.10)

$$J_2 \le NM_1^2 - \tfrac12 \sum_{k\in\mathcal{K}} R_k,$$

where

$$R_k = (a_k^{\alpha_0} - 2K^2 h\sqrt{a_k^{\alpha_0}})v_r^-\,\Delta_{h,\ell_k}\, v_r, \quad \mathcal{K} := \left\{k : a_k^{\alpha_0} - 2K^2 h\sqrt{a_k^{\alpha_0}} < 0\right\}.$$

Notice that for $k \in \mathcal{K}$ we have $a_k^{\alpha_0} < 4K^4h^2$ and hence

$$|R_k| \le 4K^4h^2|v_r^-\,\Delta_{h_k,\ell_k} v_r| \le NM_1^2$$

with a constant N depending only on K and d_1. Thus $J_2 \le NM_1^2$ and hence by (4.12) and (4.13) we get

$$m^{\alpha_0}(\nu + c^{\alpha_0})V \le NM_1\left(e^{c_oT'} + M_0 + m^{\alpha_0}M_1\right) + \xi v_r^- \tfrac{\varepsilon}{h_r},$$

where N denotes constants depending only on K and d_1. By (4.5) we have $M_1 \le \mu + V^{1/2}$, where

$$\mu := \sup_{\partial_\varepsilon Q, r} |v_r| \le e^{c_0T'}\bar\mu, \quad \bar\mu = |\sup_{\partial_\varepsilon Q, r} \delta_{h_r,l_r}\, u|.$$

Set

$$\bar M_0 = |u|_{0,Q} \ge e^{-c_0T'}M_0, \quad \bar V = e^{-2c_0T'}V.$$

Then, using Young's inequality, we obtain

$$m^{\alpha_0}(\nu + c^{\alpha_0})\bar V \le N(\bar\mu + \bar V^{1/2})\left(1 + \bar M_0 + \bar\mu + m^{\alpha_0}\bar V^{1/2}\right) + e^{-c_0T'}v_r^- \tfrac{\bar\varepsilon}{h_r}$$
$$\le N^*\left(1 + \bar M_0^{\,2} + \bar\mu^2 + m^{\alpha_0}\bar V\right) + \frac{\rho}{2}\bar V + e^{-c_oT'}v_r^- \tfrac{\varepsilon}{h_r}. \quad (4.14)$$

Assume that for c_o

$$\lambda + \nu(c_o) \ge 1 + N^*.$$

Then (4.14) yields

$$m^{\alpha_0}\,(1 + c^{\alpha_0} - \lambda)\,\bar V \le N^*\left(1 + \bar M_0^{\,2} + \bar\mu^2\right) + \frac{\rho}{2}\bar V + e^{-c_oT'}v_r^- \tfrac{\varepsilon}{h_r}.$$

Hence using condition (3.8) and then letting $\varepsilon \to 0$ we obtain

$$\bar V \le \frac{2}{\rho}N^*\left(1 + \bar M_0^{\,2} + \bar\mu^2\right),$$

that obviously yields estimate (4.4). Finally we use Remark to rewrite equation for w into equation (3.4) with Q' in place of Q, and notice that for $\gamma = (\alpha, r) \in \bar A$

$$m^\gamma|\delta_{h_r,\ell_r} f^\gamma| = m^\alpha(1+r)^{-1}|\delta_{h_r,\ell_r} f^\alpha + \tfrac{r}{m^\alpha}\delta_{h_r,\ell_r} g| \le 2K$$

for $r = \pm1, \dots, \pm(d_1+1)$. Hence the statement on w follows from that on u. □

Let us consider now (3.4)-(3.5) with $\mathcal{M}_T(\varepsilon)$ in place of Q.

Corollary 4.2. *Assume that Assumptions 3.1 through 3.3 and 4.1 with $\mathcal{M}_T(\varepsilon)$ in place of Q^0_ε hold. Let g be a bounded function on $\mathbb{R}^d$. Let u be the solution to (3.4)-(3.5) with $Q = \mathcal{M}_T(\varepsilon)$. Then there is a constant $N^* \geq 0$, depending only on d_1 and K such that for any constant $c_o \geq 0$ satisfying (4.3) we have*

$$|\delta_{\varepsilon,\pm l} u| \leq N_0 e^{c_o(T+\tau)}\big(N_1 + \sup_{\mathbb{R}^d}|g| + \max_r \sup_{\mathbb{R}^d} |\delta_{h_r,\ell_r} g|\big) \quad \text{on } \bar{\mathcal{M}}_T(\varepsilon), \tag{4.15}$$

where N_0 and N_1 are constants. The constant N_0 depends only on K, d_1 and ρ and the constant N_1 depends on K, d_1, ρ and λ, provided $\lambda > 0$, and if $\lambda = 0$ then it depends on K, d_1, ρ and T.

Proof. Let B_r denote the open ball of radius r centered at the origin in $\mathbb{R}^d$. Using Theorem 4.1 with $Q_n := \bar{\mathcal{M}}_T(\varepsilon) \cap ([0,T] \times B_n)$ in place of Q for any integer $n \geq 1$ we have

$$|\delta_{\varepsilon,\pm l} u| \leq N e^{c_o(T+\tau)}\big(1 + \max_{Q_n}|u| + \max_r \max_{\partial_\varepsilon Q_n} |\delta_{h_r,\ell_r} u|\big), \quad \text{on } Q_n,$$

where N is a constant depending only on d_1 and K. In addition to the assumptions assume that for all $\alpha \in A$ the functions f^α and g vanish outside of a fixed ball of radius R centered at the origin in $\mathbb{R}^d$. Set $\partial^T_\varepsilon = \{(T,x) \in \partial_\varepsilon Q_n\}$. Then by Lemma 3.6

$$\lim_{n\to\infty} \sup_{k,\partial_\varepsilon Q_n \setminus \partial^T_\varepsilon Q_n} (|\,\delta_{h,\ell_k} u| + |\delta_{\varepsilon,\pm l} u|) = 0.$$

Hence on $\bar{\mathcal{M}}_T(\varepsilon)$

$$|\delta_{\varepsilon,\pm l} u| \leq N e^{c_o T'}\big(1 + \sup_{\bar{\mathcal{M}}_T(\varepsilon)}|u| + \max_r \sup_{\mathbb{R}^d} |\delta_{h_r,\ell_r} g|\big). \tag{4.16}$$

Let us now remove the additional assumption on f^α and g. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative function such that $\eta \leq 1$, $|D\eta| \leq 1$ on the whole $\mathbb{R}^d$ and $\eta(x) = 1$ for $|x| \leq 1$. For each integer $n \geq 1$ define

$$f^\alpha_n(t,x) = \eta(n^{-1}x) f^\alpha(t,x), \quad g_n(x) = \eta(n^{-1}x) g(x), \quad t \geq 0,\ x \in \mathbb{R}^d.$$

Then clearly

$$\lim_{n\to\infty} (\sup_\alpha m^\alpha |f^\alpha - f^\alpha_n| + |g - g_n|) = 0 \quad \text{on } \bar{H}_T,$$

$$|f^\alpha_n| \leq |f^\alpha|, \quad |\delta_{h_r,\ell_r} f^\alpha_n| \leq |\ell_r| \sup_{\bar{H}_T} |f^\alpha| + |\delta_{h_r,\ell_r} f^\alpha|,$$

$$|g_n| \leq |g|, \quad |\delta_{h_r,\ell_r} g_n| \leq |\ell_r| \sup_{\mathbb{R}^d} |g| + |\delta_{h_r,\ell_r} g|. \tag{4.17}$$

Let u_n be the solution to (3.4)-(3.5) with $Q = \bar{\mathcal{M}}_T(\varepsilon)$ and with f^α_n and g_n in place of f^α and g, respectively. Then from (4.16) and (4.17) for all $n \in \mathbb{N}$,

$$|\delta_{\varepsilon,\pm l} u_n| \leq N e^{c_o(T+\tau)}\big(1 + \sup_{\bar{\mathcal{M}}_T(\varepsilon)} |u_n| + K \sup_{\mathbb{R}^d}|g| + \max_r \sup_{\mathbb{R}^d} |\delta_{h_r,\ell_r} g|\big).$$

Hence estimating $\sup_{\bar{\mathcal{M}}_T(\varepsilon)} |u_n|$ by using Corollary 3.11 and then letting $n \to \infty$ by using Lemma 3.12 we get estimate (4.15). □

Assumption 4.2. For all $\alpha \in A$, $t \geq 0$ and $x, y \in \mathbb{R}^d$

$$|b_k^\alpha(t,x) - b_k^\alpha(t,y)| \leq K|x-y|, \quad m^\alpha|c^\alpha(t,x) - c^\alpha(t,y)| \leq K|x-y|,$$
$$m^\alpha|f^\alpha(t,x) - f^\alpha(t,y)| \leq K|x-y|,$$
$$|\sqrt{a^\alpha(t,x)} - \sqrt{a^\alpha(t,y)}| \leq K|x-y|. \tag{4.18}$$

Theorem 4.3. *Let Assumptions 3.1 through 3.3 and Assumption 4.2 hold. Assume that g is a Borel function on $\bar{H}_T$ such that*

$$\sup_{\bar{H}_T} |g| \leq K, \quad |g(t,x) - g(ty)| \leq K|x-y| \quad \textit{for all } t \in [0,T],\ x, y \in \mathbb{R}^d.$$

Then there is a constant $N^* \geq 0$ such that for any constant $c_o \geq 0$ satisfying (4.3) *for the solution u of* (3.1)-(3.2) *and the solution w of* (3.19)-(3.20) *we have*

$$|u(t,x) - u(t,y)| + |w(t,x) - w(t,y)| \leq Ne^{c_o(T+\tau)}|x-y|, \tag{4.19}$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$, where N is a constant, that depends only on K, d_1, ρ and λ, if $\lambda > 0$. If $\lambda = 0$ then N depends on K, d_1, ρ and T.

Proof. To prove (4.18) let (t,x) and (t,y) be fixed elements of $\bar{H}_T$. We may assume that $t < T$. Moreover, by making a suitable shift in the argument of the functions, we may assume that $(t,x) \in \mathcal{M}_T$. If $|x-y| \geq K$, then estimate (4.19) holds by virtue of Corollary 3.11. Assume that $|x-y| < K$. Set $\ell = (y-x)/|x-y|$, $\ell_{\pm(d_1+1)} = \pm\ell$ and $\varepsilon = |x-y|/n$, where n is the smallest positive integer such that $|x-y|/n \leq Kh$. Then

$$|u(t,x) - u(t,y)| \leq \varepsilon \sum_{j=0}^{n-1} |\delta_{\varepsilon,\ell} u(t, x + j\varepsilon\ell)|$$
$$\leq n\varepsilon \sup_{\bar{\mathcal{M}}_T(\varepsilon)} |\delta_{\varepsilon,\ell} u| = |x-y| \sup_{\bar{\mathcal{M}}_T(\varepsilon)} |\delta_{\varepsilon,\ell} u|.$$

Hence we can finish the proof by using Corollary 4.2 if we show that Assumption 4.1 with $\mathcal{M}_T(\varepsilon)$ in place of Q^0_ε holds. It is easy to see that condition (4.1) is satisfied with K^2 in place of K. To verify condition (4.2) notice that for any $r = \pm 1, \cdots \pm (d_1+1)$, ℓ_r and $(t,z) \in \bar{\mathcal{M}}_T(\varepsilon)$

$$|\delta_{h_r,\ell_r} a_k^\alpha(t,z)| = h_r^{-1} 2(|a_k^\alpha(t,z)|^{1/2} - |a_k^\alpha(t, z + h_r\ell_r)|^{1/2})$$
$$+ h_r^{-1}\big(|a_k^\alpha(t, z + h_r\ell_r)|^{1/2} - |a_k^\alpha(t, z + h_r\ell_r)|^{1/2}\big)^2$$
$$\leq K^2 |a_k^\alpha(t,z)|^{1/2} + h_r K^4 \leq K' |a_k^\alpha(t,z)|^{1/2} + K'h$$

with $K' := 1 + K^4$. The proof is complete. □

Now we investigate the dependence of the solution to (3.4)-(3.5) on the data. Therefore together with a_k^α, b_k^α, c^α, f^α we consider also functions $\hat{a}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$, $\hat{f}^\alpha$ defined on H_∞ for each $\alpha \in A$.

Assumption 4.3. Assumptions 3.1 through 3.3 and Assumption 4.1 with $\mathcal{M}_T(0) = \mathcal{M}_T$ and $r = \pm 1, \ldots, \pm d_1$ in place of Q^0_ε and $r = \pm 1, \ldots, \pm(d_1+1)$, respectively, hold for a_k^α, b_k^α, c^α and f^α and also for $\hat{a}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$ and $\hat{f}^\alpha$ in place of a_k^α, b_k^α, c^α and f^α, respectively, with the same function m^α and constant $\lambda \geq 0$.

If Assumption 4.3 holds and g and $\hat{g}$ are bounded functions on $\mathbb{R}^d$, then by Theorem 3.4 we have, in particular, the existence of a unique bounded solution of (3.1)-(3.2) with a_k^α, b_k^α, c^α, f^α and g and also with $\hat{a}_k^\alpha$, $\hat{b}_k^\alpha$, $\hat{c}^\alpha$ $\hat{f}^\alpha$ and $\hat{g}$ in place of a_k^α, b_k^α, c^α, f^α and g, respectively. We denote these solutions by u and $\hat{u}$, respectively.

Lemma 4.4. *Let Assumption 4.3 hold. Let g and $\hat{g}$ be bounded functions on $\mathbb{R}^d$. Let $\varepsilon \in (0, Kh]$ be a constant and assume that for all $\alpha \in A$*

$$\begin{aligned} |b_k^\alpha - \hat{b}_k^\alpha| + m^\alpha |f^\alpha - \hat{f}^\alpha| + m^\alpha |c^\alpha - \hat{c}^\alpha| &\le K\varepsilon, \\ |a_k^\alpha - \hat{a}_k^\alpha| \le K\varepsilon\sqrt{a_k^\alpha \wedge \hat{a}_k^\alpha} &+ K\varepsilon h, \end{aligned} \tag{4.20}$$

on $\mathcal{M}_T$. Then there is a constant N^* depending on K and d_1 such that for any constant $c_o \ge 0$ satisfying (4.3) we have

$$\begin{aligned} |u - \hat{u}| \le \varepsilon N_0 e^{c_0(T+\tau)} \big(N_1 + \sup_{\mathbb{R}^d}(|g| + |\hat{g}| &+ \max_k |\delta_{h,\ell_k} g| \\ + \max_k |\delta_{h,\ell_k} \hat{g}| &+ \varepsilon^{-1}|g - \hat{g}|)\big) \end{aligned}$$

on $\bar{\mathcal{M}}_T$, where N_0 and N_1 are constants. The constant N_0 depends on K, d_1 and ρ. The constant N_1 depends on K, d_1, ρ and λ, provided $\lambda > 0$, and it depends on K, d_1, ρ and T when $\lambda = 0$.

Proof. We follow the idea of [14] to obtain this lemma from the gradient estimate (4.15). We consider $\mathbb{R}^d$ as a subspace $\mathbb{R}^d \times \{0\}$ of $\mathbb{R}^d \times \mathbb{R}$, and the vectors ℓ_k are identified with $(\ell_k, 0) \in \mathbb{R}^{d+1}$ for $k = \pm 1, \cdots \pm d_1$. Let $(t, x) = (t, x', x^{d+1}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$. Let $\ell = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$. Set $\ell_{\pm(d_1+1)} = \pm \ell$, $\delta_{h_r, \ell_r} = \delta_{h, \ell_k}$, for $r = k = \pm 1 \ldots, \pm d_1$, $\delta_{h_r, \ell_r} = \delta_{\varepsilon, \ell_{\pm(d_1+1)}}$ for $r = \pm(d_1 + 1)$, and

$$\bar{\mathcal{M}}_T(\varepsilon) = \bar{\mathcal{M}}_T \times \{0, \pm\varepsilon, \pm 2\varepsilon, \ldots\}, \quad \mathcal{M}_T(\varepsilon) := \bar{\mathcal{M}}_T(\varepsilon) \cap ([0, T) \times \mathbb{R}^d \times \mathbb{R}).$$

Let

$$\tilde{a}_k^\alpha(t, x', x^{d+1}) = \begin{cases} a_k^\alpha(t, x') & \text{if} \quad x^{d+1} > 0, \\ \hat{a}_k^\alpha(t, x') & \text{if} \quad x^{d+1} \le 0, \end{cases}$$

and define $\tilde{b}_k^\alpha, \tilde{c}_k^\alpha$, $\tilde{f}^\alpha$, $\tilde{g}$ and $\tilde{u}$ similarly. Then $\tilde{u}$ satisfies (3.4)-(3.5) with $\mathcal{M}_T(\varepsilon)$, $\tilde{a}_k^\alpha, \tilde{b}_k^\alpha, \tilde{c}^\alpha$, $\tilde{f}^\alpha$ and $\tilde{g}$ in place of Q, a_k^α, b_k^α, c^α, f^α and g, respectively. To apply Corollary 4.2 to $\tilde{u}$ we need to check Assumption 4.1 with $\mathcal{M}_T(\varepsilon)$, $\tilde{a}_k^\alpha, \tilde{b}_k^\alpha, \tilde{c}^\alpha$ and $\tilde{f}^\alpha$ in place of Q, a_k^α, b_k^α, c^α and f^α, respectively. Clearly this assumption with $r = \pm 1, \ldots, \pm d_1$ holds by virtue of Assumption 4.3. Since $\delta_{\varepsilon, -\ell} = -T_{\varepsilon, -\ell}\delta_{\varepsilon, \ell}$, we need only show that it holds also for $r = (d + 1)$. To this end notice that

$$\delta_{\varepsilon, \ell}\, \tilde{\psi}(t, x) = \begin{cases} 0 & \text{if} \quad x^{d+1} \ne 0, \\ \varepsilon^{-1}(\psi(t, x') - \hat{\psi}(t, x')) & \text{if} \quad x^{d+1} = 0 \end{cases}$$

with a_k^α, b^α, c^α and f^α in place of ψ. Moreover, due to (4.20)

$$\varepsilon^{-1}|a_k^\alpha(t, x') - \hat{a}_k^\alpha(t, x')| \le K\sqrt{\tilde{a}_k^\alpha(t, x', 0)} + Kh.$$

Thus $|\,\delta_{\varepsilon, \ell}\, \tilde{b}| \le K$, $m^\alpha |\,\delta_{\varepsilon, \ell}\, \tilde{c}| \le K$, $m^\alpha |\,\delta_{\varepsilon, \ell}\, \tilde{f}| \le K$ on $\mathcal{M}_T(\varepsilon)$, and

$$|\,\delta_{\varepsilon, \ell}\, \tilde{a}_k^\alpha| \le K\sqrt{\tilde{a}_k^\alpha} + Kh \quad \text{on } \mathcal{M}_T(\varepsilon).$$

Hence we get the lemma by using Corollary 4.2. $\square$

Theorem 4.5. *Let Assumptions 3.1 through 3.3 and Assumption 4.1 hold for a^α_k, b^α_k c^α and f^α and also for $\hat a^\alpha_k$, $\hat b^\alpha_k$, $\hat c^\alpha$ and $\hat f^\alpha$ in place of a^α_k, b^α_k c^α and f^α, respectively. Let g and $\hat g$ be bounded functions on $\bar H_T$ such that for all $t\in[0,T]$, $x,y\in\mathbb{R}^d$*

$$|g(t,x)|+|\hat g(t,x)|\le K,\quad |g(t,x)-g(t,y)|+|\hat g(t,x)-\hat g(t,y)|\le K|x-y|.$$

Set

$$\varepsilon=\sup_{\mathcal{M}_T,A,k}\Big(|\sigma^\alpha_k-\hat\sigma^\alpha_k|+|b^\alpha_k-\hat b^\alpha_k|+m^\alpha|c^\alpha-\hat c^\alpha|+m^\alpha|f^\alpha-\hat f^\alpha|+|g-\hat g|\Big),$$

where $\sigma^\alpha_k=\sqrt{a^\alpha_k}$, $\hat\sigma^\alpha_k=\sqrt{\hat a^\alpha_k}$. Assume that u and $\hat u$ satisfy (3.4)-(3.5), *w and $\hat w$ satisfy* (3.6)-(3.7) *with $\mathcal{M}_T$ in place of Q, and σ,b,c,f,g and $\hat\sigma,\hat b,\hat c,\hat f,\hat g$, in place of σ,b,c,f,g, respectively. Then there is a constant N^* depending on K, d_1 and ρ such that for any constant $c_o\ge 0$ satisfying* (4.3) *we have*

$$|u-\hat u|\le Ne^{c_0(T+\tau)}\varepsilon,\quad |w-\hat w|\le Ne^{c_0(T+\tau)}\varepsilon\quad\text{on }\bar{\mathcal{M}}_T,\tag{4.21}$$

where N is a constant depending on K, d_1, ρ and λ, provided $\lambda>0$. If $\lambda=0$ then N depends on K, d_1, ρ and T.

Proof. Consider first the case $\varepsilon\in(0,h]$. Then

$$|\sigma^\alpha_k-\hat\sigma^\alpha_k|\le\varepsilon,\quad |\sigma^\alpha_k-\hat\sigma^\alpha_k|^2\le\varepsilon h,$$

and using the identity

$$|a^2-b^2|=(a+b)|a-b|=2(a\wedge b)|a-b|+|a-b|^2,$$

valid for any nonnegative numbers a and b, we get

$$|a^\alpha_k-\hat a^\alpha_k|=2(|\sigma^\alpha_k|\wedge|\hat\sigma^\alpha_k|)|\sigma^\alpha_k-\hat\sigma^\alpha_k|+|\sigma^\alpha_k-\hat\sigma^\alpha_k|^2\le 2\varepsilon\sqrt{a^\alpha_k\wedge\hat a^\alpha_k}+\varepsilon h.$$

Hence by Lemma 4.4, $|u-\hat u|\le\varepsilon Ne^{c_0(T+\tau)}$ on $\bar{\mathcal{M}}_T$. Now consider the case $\varepsilon>h$. For $\theta\in[0,1]$, let u^θ be the solution of

$$\sup_\alpha m^\alpha(\delta_\tau u^\theta+a^{\theta\alpha}_k\Delta_{h,\ell_k}u^\theta+b^{\theta\alpha}_k\delta_{h,\ell_k}u^\theta-c^{\theta\alpha}u^\theta+f^{\theta\alpha})=0\text{ on }\mathcal{M}_T$$
$$g^\theta=u^\theta\text{ on }\{(T,x)\in\bar{\mathcal{M}}_T\},$$

where

$$(\sigma^{\theta\alpha}_k,b^{\theta\alpha}_k,c^{\theta\alpha},f^{\theta\alpha},g^\theta)=(1-\theta)(\sigma^\alpha_k,b^\alpha_k,c^\alpha,f^\alpha,g)+\theta(\hat\sigma^\alpha_k,\hat b^\alpha_k,\hat c^\alpha,\hat f^\alpha,\hat g)$$

and $a^{\theta\alpha}_k=(1/2)|\sigma^{\theta\alpha}_k|^2$. For any $\theta_1,\theta_2\in[0,1]$,

$$|\sigma^{\theta_1\alpha}_k-\sigma^{\theta_2\alpha}_k|+|b^{\theta_1\alpha}_k-b^{\theta_2\alpha}_k|+|c^{\theta_1\alpha}-c^{\theta_2\alpha}|$$
$$+m^\alpha|f^{\theta_1\alpha}-f^{\theta_2\alpha}|+|g^{\theta_1}-g^{\theta_2}|\le|\theta_1-\theta_2|\varepsilon.$$

Hence if θ_1,θ_2 satisfy $|\theta_1-\theta_2|\varepsilon\le h$, then, thanks to the first part of the proof, with u^{θ_1} and u^{θ_2} playing the roles of u and $\hat u$, respectively,

$$|u^{\theta_1}-u^{\theta_2}|\le N|\theta_1-\theta_2|\varepsilon e^{c_o(T+\tau)}.$$

Set $\theta_i:=i/m$ for $i=0,1,\dots,m$ for an integer $m\ge1$ such that $\varepsilon/m\le h$. Then

$$|\hat u-u|\le\sum_{i=0}^{m-1}|u^{\theta_{i+1}}-u^{\theta_i}|\le N\sum_{i=0}^{m-1}|\theta_{i+1}-\theta_i|\varepsilon e^{c_oT'}=N\varepsilon e^{c_o(T+\tau)},$$

that proves (4.21) for u and $\hat{u}$. Hence by using Remark 3.1 to rewrite equation (3.6) we get (4.21) also for w and $\hat{w}$. □

5. Some properties of the reward functions

Let A be a separable metric space. Let $\sigma = \sigma^\alpha(t,x)$ and $\beta = \beta^\alpha(t,x)$ be some Borel functions of $(\alpha,t,x) \in A \times [0,\infty) \times \mathbb{R}^d$ with values in $\mathbb{R}^{d\times d'}$ and $\mathbb{R}^d$, respectively. Let $\alpha = (\alpha_t)_{t\geq 0}$ be a progressively measurable process with values in A, such that for every $s \in [0,T)$ and $x \in \mathbb{R}^d$ there is a solution $x_t = \{x_t^{\alpha,s,x} : t \in [0,T-s]\}$ of equation (2.3).

Let $f = f^\alpha(t,x)$, $c = c^\alpha(t,x) \geq \lambda$ and $g = g(t,x)$ be Borel functions of $(\alpha,t,x) \in A \times [0,\infty) \times \mathbb{R}^d$ and of $(t,x) \in [0,\infty) \times \mathbb{R}^d$, respectively, where $\lambda \geq 0$ is some constant. Set

$$v^\alpha(s,x) = \mathbb{E}\int_0^{T-s} f^{\alpha_t}(s+t, x_t^{\alpha,s,x})e^{-\varphi_t}\,dt + \mathbb{E}g(T, x_{T-s}^{\alpha,s,x})e^{-\varphi_{T-s}},$$

$$w^{\alpha,\tau}(s,x) = \mathbb{E}\int_0^{\tau} f^{\alpha_t}(s+t, x_t^{\alpha,s,x})e^{-\varphi_t}\,dt + \mathbb{E}g(s+\tau, x_\tau^{\alpha,s,x})e^{-\varphi_\tau}, \quad (5.1)$$

$$\varphi = \varphi_t^{\alpha,x} = \int_0^t c^{\alpha_u}(s+u, x_u^{\alpha,s,x})\,du$$

for $s \in [0,T]$, $x \in \mathbb{R}^d$, for the process $\alpha = (\alpha_t)$ and for a fixed stopping times τ with values in $[0,T-s]$.

Lemma 5.1. *Assume that there exists a constant K such that $|g| \leq K$ on $\bar{H}_T$ and*

$$|f^\alpha(t,x)| \leq K(1 + c^\alpha(t,x)) \quad (5.2)$$

for all $\alpha \in A$, $t \geq 0$ and $x \in \mathbb{R}^d$. Then for $u := v^\alpha, w^{\alpha,\tau}$ we have

$$|u| \leq K(2+N) \quad \text{on } \bar{H}_T,$$

where $N = (1-\exp(-\lambda T))/\lambda$ if $\lambda > 0$, and $N = T$ if $\lambda = 0$.

Proof. Notice that

$$\int_0^{T-s} c^\alpha(s+t, x_t)e^{-\varphi_t}\,dt = 1 - e^{-\varphi_{T-s}} \leq 1.$$

Hence

$$|u(s,x)| \leq K\mathbb{E}\int_0^{T-s}(1 + c^\alpha(s+t,x_t))e^{-\varphi_t}\,dt + K \leq K(2+N).$$

□

Assumption 5.1. There exist a Borel function $m : A \to (0,1]$ and constants $\rho > 0$, $K \geq 0$ and L such that for all $\alpha \in A$, $t \in [0,T]$ and $x, y \in \mathbb{R}^d$

$$m^\alpha(1 + c^\alpha(t,x) - \lambda) \geq \rho, \quad |m^\alpha f^\alpha(t,x)| \leq K, \quad (5.3)$$

$$m^\alpha|f^\alpha(t,x) - f^\alpha(t,y)| \leq K|x-y| \quad (5.4)$$

$$|c^\alpha(t,x) - c^\alpha(t,y)| \leq K|x-y|, \quad (5.5)$$

$$|g(t,x) - g(t,y)| \leq K|x-y|, \quad (5.6)$$

$$(x-y)(\beta^\alpha(t,x) - \beta^\alpha(t,y)) + \tfrac{1}{2}|\sigma^\alpha(t,x) - \sigma^\alpha(t,y)|^2 \leq L|x-y|^2. \quad (5.7)$$

Remark 5.2. Notice that condition (5.3) implies condition (5.2) of Lemma 5.1, with K/ρ in place of K in (5.2). Clearly, if β^α and σ^α are Lipschitz continuous in $x \in \mathbb{R}^d$, with Lipschitz constant $L/2$, independent of $\alpha \in A$ and $t \in [0,T]$, then the *monotonicity condition* (5.7) is satisfied.

Lemma 5.3. *Let Assumption 5.1 hold. Assume*

$$|g| \le K \quad \text{on } \bar{H}_T. \tag{5.8}$$

Then for $u := v^\alpha, w^{\alpha,\tau}$ we have

$$|u(s,x) - u(s,y)| \le N|x-y| \quad \text{for all } s \in [0,T] \text{ and } x,y \in \mathbb{R}^d,$$

where N is a constant depending only on K, ρ and T. If $\lambda \ge |L| + 2$, then N depends only on K and ρ.

Proof. Clearly, $|u(s,x) - u(s,y)| \le \sum_{k=1}^4 I_k$ with

$$I_1 = \int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{\alpha,s,x})||e^{-\varphi_t^{\alpha,s,x}} - e^{-\varphi_t^{\alpha,s,y}}|dt,$$

$$I_2 = \mathbb{E}\int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{\alpha,s,x}) - f^{\alpha_t}(s+t, x_t^{\alpha,s,y})|e^{-\varphi_t^{\alpha,s,y}}dt,$$

$$I_3 = \sup_{\tau \in \mathfrak{T}(T-s)} \mathbb{E}\{|g(s+\tau, x_\tau^{\alpha,s,x})||e^{-\varphi_\tau^{\alpha,s,x}} - e^{-\varphi_\tau^{\alpha,s,y}}|\},$$

$$I_4 = \sup_{\tau \in \mathfrak{T}(T-s)} \mathbb{E}\{|g(s+\tau, x_\tau^{\alpha,s,x}) - g(s+\tau, x_\tau^{\alpha,s,y})|e^{-\varphi_\tau^{\alpha,s,y}}\}.$$

By (5.5) and (5.3)

$$I_1 \le \mathbb{E}\int_0^{T-s} |f^{\alpha_t}(s+t, x_t^{\alpha,s,x})||\varphi_t^{\alpha,s,x} - \varphi_t^{\alpha,s,y}|e^{-\min(\varphi_t^{\alpha,s,x},\varphi_t^{\alpha,s,y})}dt$$

$$\le K^2\mathbb{E}\int_0^{T-s} te^{-(\lambda-1)t}\tfrac{1}{m^{\alpha_t}}\sup_{r\le t}|x_r^{\alpha,s,x} - x_r^{\alpha,s,y}|e^{-\int_0^t \frac{\rho}{m^{\alpha_u}}\,du}\,dt$$

$$\le N_1 \sup_{t \le T-s} e^{-N_0 t}\mathbb{E}\sup_{r\le t}|x_r^{\alpha,s,x} - x_r^{\alpha,s,y}|,$$

for any constant $N_0 \ge 0$, where $N_1 = K^2(e(\lambda-1-N_0))^{-1}\rho^{-1}$ when $\lambda \ge 1+N_0$, and N_1 depends on ρ, K N_0 and T when $\lambda \in [0, 1+N_0]$. By (5.4) and (5.3)

$$I_2 \le K\mathbb{E}\int_0^{T-s} (m^{\alpha_t})^{-1}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|e^{-\varphi_t^{\alpha,s,y}}dt$$

$$\le K\mathbb{E}\left(\sup_{t\le T} e^{-N_0 t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|\int_0^{T-s} e^{-(\lambda-1-N_0)t}\tfrac{1}{m^{\alpha_t}}e^{-\int_0^t \frac{\rho}{m^{\alpha_u}}\,du}dt\right)$$

$$\le N_2\mathbb{E}\sup_{t\le T-s} e^{-N_0 t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|,$$

for every constant $N_0 \ge 0$, where $N_2 = K/\rho$ when $\lambda \ge 1+N_0$, and N_2 depends on K, ρ, N_0 and T when $\lambda < 1 + N_0$. Due to conditions (5.8), (5.5), $c^\alpha \ge \lambda$ and (5.6) we have

$$I_3 \le K \sup_{\tau \in \mathfrak{T}(T-s)} \mathbb{E}|e^{-\varphi_\tau^{\alpha,s,x}} - e^{-\varphi_\tau^{\alpha,s,y}}| \le K\mathbb{E}\sup_{t \le T-s} e^{-\lambda t}\int_0^t |x_r^{\alpha,s,x} - x_r^{\alpha,s,y}|\,dr$$

$$\leq K \int_0^{T-s} \mathbb{E}e^{-\lambda r}|x_r^{\alpha,s,x} - x_r^{\alpha,s,y}|\,dr \leq N_3 \sup_{t\leq T-s} \mathbb{E}e^{-N_0 t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|$$

for every constant $N_0 \geq 0$, where N_3 depends only on K if $\lambda > N_0 + 1$ and N_3 depends on K, N_0 and T if $\lambda \leq N_0 + 1$. Similarly,

$$I_4 \leq K\mathbb{E} \sup_{t\leq T-s} e^{-\lambda t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| \leq N_4 \mathbb{E} \sup_{t\leq T-s} e^{-N_0 t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|$$

for any $N_0 \geq 0$, where $N_4 = K\exp((N_0 - \lambda)T)$. Consequently,

$$|u(s,x) - u(s,y)| \leq N\mathbb{E} \sup_{t\leq T-s} e^{-N_0 t}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| \tag{5.9}$$

for every $N_0 \geq 0$. The constant N depends only on K and ρ, if $\lambda \geq N_0 + 2$, and it depends on K, ρ, N_0 and T if $\lambda < 2 + N_0$. Using Itô's formula and condition (5.7), we have

$$e^{-2Lt}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}|^2 \leq |x-y|^2 + M_t$$

almost surely for all $t \in [0, T-s]$, where M is a local martingale. Thus

$$\mathbb{E}e^{-2L\tau}|x_\tau^{\alpha,s,x} - x_\tau^{\alpha,s,y}|^2 \leq |x-y|^2$$

for all stopping times $\tau \leq T - s$, that yields

$$\mathbb{E} \sup_{t\leq T-s} e^{-Lt}|x_t^{\alpha,s,x} - x_t^{\alpha,s,y}| \leq 3|x-y|$$

by virtue of Lemma 3.2 from [5]. Combining this with estimate (5.9) we finish the proof of the lemma. □

Assume that $A = \cup_{n=1}^\infty A_n$ for an increasing sequence of Borel sets A_n of A such that Assumptions 2.1 and 2.2 hold with A_n. Then the reward functions v^α and $w^{\alpha,\tau}$ are well-defined on $\bar{H}_T$ for every $\alpha \in \mathfrak{A} = \cup_{n=1}^\infty \mathfrak{A}_n$, where $\mathfrak{A}_n$ denotes the set of progressively measurable processes $\alpha = (\alpha_t)_{t\geq 0}$ taking values in A_n. Thus we can define the optimal reward functions

$$v(s,x) = \sup_{\alpha\in\mathfrak{A}} v^\alpha(s,x), \quad w(s,x) = \sup_{\alpha\in\mathfrak{A}} \sup_{\tau\in\mathfrak{T}(T-s)} w^{\alpha,\tau}(s,x)$$

for every $(s,x) \in [0,T] \times \mathbb{R}^d = \bar{H}_T$. Recall the notation $H_T := [0,T) \times R^d$,

$$L^\alpha = \sigma_{ik}^\alpha \sigma_{jk}^\alpha D_i D_j + \beta_i^\alpha D_i + c^\alpha,$$

and let $C^{1,2}(\bar{H}_T)$ denote the set of functions $\psi = \psi(t,x)$ whose first derivative in t and second order derivatives in x are continuous functions on $\bar{H}_T$. The following lemma formulates an important property of smooth supersolutions and subsolutions to Bellman equations.

Lemma 5.4. *Let Assumptions 2.1 and 2.2 hold. Assume that σ, β are continuous in $\alpha \in A$. Assume, moreover, that f and c are continuous in (α, x) and are continuous in x, uniformly in $\alpha \in A$, for each $t \in [0,T]$. Let $S \in (0,T]$ and $\psi \in C^{1,2}(\bar{H}_S)$ such that for some constants K and $q \geq 0$*

$$|\psi(t,x)| \leq K(1 + |x|^q) \quad \text{for all } (t,x) \in H_S. \tag{5.10}$$

Let Q be a domain contained in H_S. Denote its boundary by ∂Q. Then the following statements hold:

(i) *Let*
$$\frac{\partial}{\partial t}\psi + L^\alpha\psi + f^\alpha \le 0 \quad \text{on } Q, \text{ for all } \alpha \in A. \tag{5.11}$$
Then
$$v \le \psi + \sup_{\partial Q}[v-\psi]_+ \quad \text{on } \bar{Q}. \tag{5.12}$$
In addition to (5.11) *let* $g \le \psi$ *on* Q. *Then* (5.12) *holds also for* w *in place of* v.

(ii) *Let*
$$\frac{\partial}{\partial t}\psi + L^\alpha\psi + f^\alpha \ge 0 \quad \text{on } Q, \text{ for some } \alpha \in A. \tag{5.13}$$
Then
$$v \ge \psi - \sup_{\partial Q}[v-\psi]_- \quad \text{and} \quad w \ge \psi - \sup_{\partial Q}[w-\psi]_- \ \text{on} \quad \bar{Q}. \tag{5.14}$$

Proof. This lemma follows from Lemma 6.1.2 and Theorem 6.1.5 from [10]. For the convenience of the reader we give a more detailed proof here. Set $v_n = \sup_{\alpha\in\mathfrak{A}_n} v^\alpha$ for integers $n \ge 1$. Then by Theorem 3.1.5 in [10], the polinomial growth condition (5.10) holds for v_n in place of ψ, with some constants K and q depending on n, and v_n is continuous on $\bar{H}_T$. Set
$$\tau_Q = \inf\{t \ge 0 : (s+t, x_t) \notin Q\}, \quad \tau_Q^R = \inf\{t \ge 0 : |x_t| \ge R\} \wedge \tau_Q$$
for $R > 0$. By Bellman's principle (Theorem 2.3.6 from [10]), for $(s,x) \in Q$, integer $n \ge 1$, stopping time $\tau = \tau_Q^R$, for any $\varepsilon > 0$ there is a strategy $(\alpha_t) \in \mathfrak{A}_n$ such that
$$v_n(s,x) \le \varepsilon + I_n^{(\alpha)}(s,x), \tag{5.15}$$
$$I_n^{(\alpha)}(s,x) := \mathbb{E}_{s,x}^\alpha \left(\int_0^\tau f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + v_n(s+\tau, x_\tau)e^{-\varphi_\tau} \right), \tag{5.16}$$
where, as before, to ease notation we use x_t in place of $x_t^{\alpha,s,x}$. Using condition (5.11) and applying Itô's formula to $\psi(s+t,x_t)e^{-\varphi_t}$ we have
$$I_n^{(\alpha)}(s,x) \le -\mathbb{E}_{s,x}^\alpha \int_0^\tau e^{-\varphi_t}\left[\tfrac{\partial}{\partial t} + L^{\alpha_t}\right]\psi(s+t,x_t)dt + \mathbb{E}_{s,x}^\alpha v_n(s+\tau,x_\tau)e^{-\varphi_\tau}$$
$$= \psi(s,x) + \mathbb{E}_{s,x}^\alpha\{(v_n(s+\tau,x_\tau) - \psi(s+\tau,x_\tau))e^{-\varphi_\tau}\}.$$
Letting here $R \to \infty$ we get
$$I_n^{(\alpha)}(s,x) \le \psi(s,x) + \mathbb{E}_{s,x}^\alpha\{(v_n(s+\tau_Q,x_{\tau_Q}) - \psi(s+\tau_Q,x_{\tau_Q}))e^{-\varphi_{\tau_Q}}\}.$$
Thus from (5.15) we have
$$v_n(s,x) \le \varepsilon + \psi(s,x) + \sup_{\partial Q}[v_n-\psi]_+ \le \varepsilon + \psi(s,x) + \sup_{\partial Q}[v-\psi]_+ .$$
Letting here $n \to \infty$ and $\varepsilon \to 0$ we get (5.12). Hence (5.12) is valid also for w in place of v, since $w = \sup_{\gamma\in\bar{\mathfrak{A}}} v^\gamma$ by virtue of Theorem 2.1, Assumptions 2.1-2.2 remain valid with $\bar{A}_n$ and $\bar{A}$ in place of A_n and A, and due to (5.11) and $\psi \ge g$ on Q,
$$\tfrac{\partial}{\partial t}\psi + L^\gamma\psi + f = \tfrac{\partial}{\partial t}\psi + L^\alpha\psi + f + r(g-\psi) \le 0 \quad \text{on } Q$$
for every $\gamma = (\alpha, r) \in \bar{A}$. To prove (ii) let $\alpha \in A$ such that (5.13) holds. Then $\alpha \in A_n$ for some $n \ge 1$, the constant strategy $\alpha_t = \alpha$ belongs to $\mathfrak{A}_n$, and by Bellman's principle
$$v_n(s,x) \ge I_n^{(\alpha)}(s,x)$$

with this strategy α, where $I_n^{(\alpha)}$ is defined by (5.16). Hence by an obvious modification of the proof of part (i) we get the first inequality in (5.14), and that yields the second inequality by virtue of Theorem 2.1, since clearly $\frac{\partial}{\partial t}\psi + L^\gamma \psi + f^\gamma \geq 0$ on Q for $\gamma = \alpha \in A \subset \bar{A}$. □

Next we want to study the regularity of v and w in $t \in [0, T]$. The following simple example shows that Assumption 5.1 does not ensure the continuity of v at $t = T$, even if σ^α and b^α are as regular as we wish.

Example 5.5. Let $A = [0, \infty)$, $f^\alpha(t, x) = \alpha$, $g(x) = 0$, $c^\alpha(t, x) = \alpha$ for $\alpha \in A$. Then Assumption 5.1 holds with $m^\alpha = (1 + \alpha)^{-1}$ and $\sigma^\alpha = 0$, $b^\alpha = 0$, and for $t \in [0, T]$, $x \in \mathbb{R}^d$

$$v(t, x) = \sup_{\alpha \in \mathfrak{A}} \mathbb{E} \int_0^{T-t} \alpha_s e^{-\int_0^s \alpha_u du} ds = \sup_{\alpha \in \mathfrak{A}} \mathbb{E} \left(1 - e^{-\int_0^{T-t} \alpha_u du}\right) = \mathbf{1}_{t<T},$$

which is not continuous at T.

6. Hölder continuity in time

Let $\sigma = \sigma^\alpha(t, x)$, $\beta = \beta^\alpha(t, x)$, $f = f^\alpha(t, x)$ and $c = c^\alpha(t, x)$ be Borel functions of $(\alpha, t, x) \in A \times \mathbb{R}_+ \times \mathbb{R}^d$, taking values in $\mathbb{R}^{d \times d'}$, $\mathbb{R}^d$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively, such that $c \geq \lambda$ for a constant $\lambda \geq 0$. Let g be a Borel function on $\mathbb{R}_+ \times \mathbb{R}^d$ with values in $\mathbb{R}$.

We make the following assumption.

Assumption 6.1. There is a constant K such that for $\psi = \sigma^\alpha, \beta^\alpha, f^\alpha, c^\alpha, g$ for all $\alpha \in A$ we have

$$|\psi(t, x) - \psi(t, y)| \leq K|x - y|, \quad |\psi(t, x)| \leq K$$

for all $t \in [0, T]$ and $x, y \in \mathbb{R}^d$.

Obviously Assumption 6.1 implies Assumptions 2.1 and 2.2, the reward functions v^α, $w^{\alpha,\tau}$, v and w are well-defined by (2.8), (2.9) and (2.7). Moreover, Assumption 5.1 holds with $m^\alpha = \rho = 1$ and $L = 2K$. Thus by Lemma 5.3 there is a constant C such that for $u := v, w$

$$|u(t, x) - u(t, y)| \leq C|x - y| \quad \text{for all } t \in [0, T] \text{ and } x, y \in \mathbb{R}^d. \tag{6.1}$$

If $\lambda \geq K + 2$, then C depends only on K, otherwise it depends on K and T. Using results from [10] and [15] one can prove the following lemma on the Hölder continuity of v and w in t.

Lemma 6.1. *Let Assumption 6.1 hold. Assume that σ, β are continuous in $\alpha \in A$. Assume, moreover, that f and c are continuous in (α, x) and are continuous in x, uniformly in $\alpha \in A$, for each $t \in [0, T]$. Then for $x_0 \in \mathbb{R}^d$ and $0 \leq t_0 \leq s_0 \leq T$ such that $|s_0 - t_0| \leq 1$, we have*

$$|v(t_0, x_0) - v(s_0, x_0)| \leq N(\nu_1 + 1)|s_0 - t_0|^{1/2}, \tag{6.2}$$

$$|w(t_0, x_0) - w(s_0, x_0)| \leq N(\nu_2 + 1)|s_0 - t_0|^{1/2} + \mu|s_0 - t_0|^{1/2}, \tag{6.3}$$

where N is a constant depending only on K, and

$$\nu_1 := \sup_{y \in \mathbb{R}^d \setminus \{x_0\}} \tfrac{|v(s_0,x_0) - v(s_0,y)|}{|x_0 - y|}, \quad \nu_2 := \sup_{y \in \mathbb{R}^d \setminus \{x_0\}} \tfrac{|w(s_0,x_0) - w(s_0,y)|}{|x_0 - y|},$$

$$\mu := \sup_{y\in\mathbb{R}^d}\sup_{0\le t<s_0} \tfrac{|g(t,y)-g(s_0,y)|}{|t-s_0|^{1/2}}. \tag{6.4}$$

Proof. We may assume $\nu_1 < \infty$, $\nu_2 < \infty$, $\mu < \infty$ and $0 \le t_0 < s_0$. Moreover, by shifting the origin we may assume $t_0 = 0$ and hence $s_0 \le 1$. To prove (6.2) define for a constant $\gamma > 0$ the function

$$\psi(t,x) = \gamma\nu_1[\xi(t)|x-x_0|^2 + \kappa_1(s_0-t)] + \kappa_2(s_0-t)$$
$$+\nu_1\gamma^{-1} + v(s_0,x_0), \quad \text{for } (t,x) \in \bar{H}_{s_0}, \tag{6.5}$$

where $\xi(t) = \exp(s_0-t)$ and $\kappa_1 > 0$, $\kappa_2 > 0$ are some constants to be chosen later. By simple calculations for any $\alpha \in A$

$$L^\alpha\psi(t,x) = \nu_1\gamma\xi(t)[2\sigma^\alpha_{ik}\sigma^\alpha_{ik}(t,x) + 2\beta^\alpha_i(t,x)(x_i-x_{0i})]$$
$$-c^\alpha(t,x)\psi(t,x) \le N_1\gamma\nu_1(1+|x-x_0|) + N_2,$$

for $(t,x) \in \bar{H}_{s_0}$, where N_1 and N_2 are constants depending only on K. Hence, choosing $\kappa_2 \ge K + N_2$, we have

$$\tfrac{\partial}{\partial t}\psi(t,x) + L^\alpha\psi(t,x) + f^\alpha(t,x)$$
$$\le \nu_1\gamma[N_1(1+|x-x_0|) - |x-x_0|^2 - \kappa_1], \tag{6.6}$$

where the right-hand side is negative for all x if κ_1 is sufficiently large, depending only on N_1. Notice that for all $x \in \mathbb{R}^d$

$$\psi(s_0,x) = \nu_1(\gamma|x-x_0|^2+\gamma^{-1}) + v(s_0,x_0) \ge \nu_1|x-x_0| + v(s_0,x_0) \ge v(s_0,x).$$

Thus applying part (i) of Lemma 5.4 with $S := s_0$ and $Q := H_{s_0}$ we obtain

$$v(t,x_0) \le \nu_1[\gamma\kappa_1(s_0-t) + \gamma^{-1}] + \kappa_2(s_0-t) + v(s_0,x_0)$$

for all $t \in [0,s_0]$ and constants $\gamma > 0$. For $t = 0$ we choose $\gamma = (\kappa_1 s_0)^{-1/2}$ to get

$$v(0,x_0) \le 2\nu_1\kappa_1^{1/2}s_0^{1/2} + \kappa_2 s_0 + v(s_0,x_0),$$

that yields

$$v(0,x_0) - v(s_0,x_0) \le N(\nu_1+1)s_0^{1/2} \tag{6.7}$$

with $N = \max(2\kappa_1^{1/2}, \kappa_2)$. To get the corresponding estimate for w, instead of (6.5) define ψ by

$$\psi(t,x) = \gamma\nu_2[\xi(t)|x-x_0|^2 + \kappa_1(s_0-t)] + \kappa_2(s_0-t)$$
$$+\nu_2\gamma^{-1} + \mu s_0^{1/2} + w(s_0,x_0). \tag{6.8}$$

Then just like before we see that for sufficiently large constants κ_1 and κ_2, depending only on K, the left-hand side of (6.6) remains negative for all $(t,x) \in \bar{H}_{s_0}$, and that

$$\psi(t,x) \ge \psi(s_0,x) \ge w(s_0,x) + \mu s_0^{1/2} \ge g(s_0,x) + \mu s_0^{1/2} \ge g(t,x)$$

for all $t \in [0,s_0]$ and $x \in \mathbb{R}^d$. Hence by part (i) of Lemma 5.4

$$w(0,x_0) \le \nu_2[\gamma\kappa_1 s_0 + \gamma^{-1}] + \kappa_2 s_0 + \mu s_0^{1/2} + w(s_0,x_0)$$

for any $\gamma > 0$, that yields

$$w(0,x_0) \le N(\nu_2+1)s_0^{1/2} + w(s_0,x_0).$$

Now we prove this inequality with $w(0,x)$ and $w(s_0,x)$ interchanged, together with inequality (6.7) with $v(0,x)$ and $v(s_0,x)$ interchanged. To this end set

$$\psi(t,x) = -\gamma\nu[\xi(t)|x-x_0|^2 + \kappa_1(s_0-t)] - \kappa_2(s_0-t) - C\gamma^{-1} + u(s_0,x_0).$$

with $u := v$ and w, and $\nu := \nu_1$ and ν_2, respectively. Notice that for large κ_2, depending only on K, we have

$$\frac{\partial}{\partial t}\psi(t,x) + L^\alpha\psi(t,x) + f^\alpha(t,x) \geq -\nu\gamma[N_1(1+|x-x_0|) - |x-x_0|^2 - \kappa_1],$$

with a constant N_1 depending on K, where the right-hand side is positive for all x if κ_1 is sufficiently large, depending only on N_1. Furthermore,

$$\psi(s_0,x) = -\nu(\gamma|x-x_0|^2 + \gamma^{-1}) + u(s_0,x_0)$$

$$\leq -\nu|x-x_0| + u(s_0,x_0) \leq u(s_0,x).$$

Hence by virtue of part (ii) of Lemma 5.4 we get

$$u(t,x_0) \geq -\nu[\gamma\kappa_1(s_0-t) + \gamma^{-1}] - \kappa_2(s_0-t) + u(s_0,x_0)$$

for all $t \in [0,s_0]$ and constant $\gamma > 0$. Choosing here $t = t_0 = 0$ and $\gamma = (\kappa_1 s_0)^{-1/2}$ we get

$$u(0,x_0) \geq -2\nu\kappa_1^{1/2} - \kappa_2 s_0^{1/2} + u(s_0,x_0) \geq -N(\nu+1)s_0^{1/2} + u(s_0,x_0)$$

with $N := \max(2\kappa_1^{1/2}, \kappa_2)$, that completes the proof of the lemma. □

Theorem 6.2. *Let Assumption 6.1 hold. Assume that σ^α, β^α, f^α are continuous in $\alpha \in A$ and that*

$$|g(s,x) - g(t,x)| \leq K|t-s|^{1/2} \quad \text{for all } t,s \in [0,T] \text{ and } x \in \mathbb{R}^d.$$

Then there is a constant N such that for $u := v, w$ we have

$$|u(s,x) - u(t,x)| \leq N|t-s|^{1/2} \quad \text{for all } t,s \in [0,T] \text{ and } x \in \mathbb{R}^d.$$

The constant N depends on K and T. Moreover, there is a constant λ_0, depending on K, such that if $\lambda \geq \lambda_0$, then N depends only on K.

Proof. We get this theorem immediately from the previous lemma by taking into account Lemmas 5.1 and 5.3. □

Now we formulate the corresponding results for the solutions $v = v_{\tau,h}$ and $w = w_{\tau,h}$ of the finite difference schemes (3.1)-(3.2) and (3.19)-(3.20), respectively, when $m^\alpha = 1$ for all $\alpha \in A$. The following lemma is proved in [15] for $u = v_{\tau,h}$.

Lemma 6.3. *Let $\tau, h \leq K$. Let Assumption 3.1 hold and assume that for $\psi := a_k^\alpha$, b_k^α, c^α, f^α and g for every $k = \pm1, \cdots \pm d_1$ and $\alpha \in A$ we have $|\psi| \leq K$ on $\bar{H}_T$. Let $(t_0,x_0) \in \bar{H}_T$ and $s_0 \in [t_0,T]$ such that $s_0 - t_0 \leq 1$ and $(s_0-t_0)/\tau$ is an integer. Then (6.2) and (6.3) hold with $v_{\tau,h}$ and $w_{\tau,h}$ in place of v and w, respectively, where the constants ν_1, ν_2 and μ are defined by (6.4) with $v_{\tau,h}$ and $w_{\tau,h}$, in place of v and w, respectively, and the constant N depends on K and d_1.*

Proof. We may assume that $s_0 > 0$ and also, by shifting the origin, that $t_0 = 0$, $x_0 = 0$ and hence that $s_0 \in (0,1)$ is an integer multiple of τ. Now we can prove the required estimates in the same way as Lemma 6.1 is proved. We need only use Corollary 3.10 with $T := s_0$ and $Q := \mathcal{M}_{s_0}$ in place of Lemma 5.4. □

Theorem 6.4. *Let $\tau, h \le K$. Let Assumption 3.1 hold and assume that for $\psi := \sqrt{a^\alpha_k}$, b^α_k, c^α, f^α and g, for every $k = \pm 1, \cdots \pm d_1$ and $\alpha \in A$ we have*

$$|\psi(t,x) - \psi(s,y)| \le K(|x-y| + |s-t|^{1/2}), \quad |\psi(t,x)| \le K$$

for all $s,t \in [0,T]$ and $x,y \in \mathbb{R}^d$. Then for $u := v_{\tau,h}$, $w_{\tau,h}$ we have

$$|u(t,x) - u(s,x)| \le N\left(|t-s|^{1/2} + \tau^{1/2}\right) \tag{6.9}$$

for all $x \in \mathbb{R}^d$ and $s,t \in [0,T]$, where N is a constant depending only on K, d_1 and T. There is a constant $\lambda_0 \ge 0$, depending only on K and d_1, such that if $\lambda \ge \lambda_0$ then N depends only on K and d_1.

Proof. For $u = v_{\tau,h}$ estimate (6.9) is proved in [15] (see Lemma 6.2 there). We get (6.9) for $u = w_{\tau,h}$ similarly, noticing that Assumptions 3.2 and 3.3 are obviously satisfied with $m^\alpha = 1$ and $\rho = 1$, and by using Lemma 6.3, Theorems 4.3, 4.5 and Corollary 3.11. □

7. SHAKING AND SMOOTHING

The *method of shaking* is introduced in [13]. Following [15] we adapt it to optimal stopping of controlled diffusion processes and to the corresponding finite difference schemes.

For $\varepsilon \in \mathbb{R}$ we set

$$A^\varepsilon = A \times [-\varepsilon^2, 0] \times \{x \in \mathbb{R}^d : |x| \le \varepsilon\}, \quad \bar{A}^\varepsilon = A^\varepsilon \times [0,\infty),$$

and identify $\alpha \in A$ with $(\alpha,0,0) \in A^\varepsilon$ and $(\alpha,\eta,\xi) \in A^\varepsilon$ with $(\alpha,\eta,\xi,0) \in \bar{A}^\varepsilon$. Thus $A \subset A^\varepsilon \subset \bar{A}^\varepsilon$.

First we shake optimal stopping and control problems. Let $\sigma = \sigma^\alpha(t,x)$, $\beta = \beta^\alpha(t,x)$, $f = f^\alpha(t,x)$ and $c = c^\alpha(t,x)$ be Borel functions of $(\alpha,t,x) \in A \times \mathbb{R} \times \mathbb{R}^d$, taking values in $\mathbb{R}^{d\times d'}$, $\mathbb{R}^d$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively, such that $c \ge \lambda$ for a constant $\lambda \ge 0$. Let g be a Borel function on $\mathbb{R} \times \mathbb{R}^d$ with values in $\mathbb{R}$.

We make the following assumption.

Assumption 7.1. There is a constant K such that for $\psi = \sigma^\alpha, \beta^\alpha, f^\alpha, g$, $c^\alpha - \lambda$, for all $\alpha \in A$ we have

$$|\psi(t,x) - \psi(s,y)| \le K(|x-y| + |s-t|^{1/2}), \quad |\psi(t,x)| \le K$$

for all $s,t \in \mathbb{R}$ and $x,y \in \mathbb{R}^d$.

For $\gamma = (\alpha,\eta,\xi,r) \in \bar{A}^\varepsilon$ we set

$$\sigma^\gamma(t,x) = \sigma^\alpha(t+\eta, x+\xi), \quad \beta^\gamma(t,x) = \beta^\alpha(t+\eta, x+\xi),$$

$$c^\gamma(t,x) = c^\alpha(t+\eta, x+\xi) + r,$$

$$f^\gamma(t,x) = f^\alpha(t+\eta, x+\xi) + rg^\varepsilon(t,x), \tag{7.1}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, where

$$g^\varepsilon(t,x) := \sup_{\eta \in [-\varepsilon^2, 0]} \sup_{\xi \in \mathbb{R}^d, |\xi| \le \varepsilon} |g(t+\eta, x+\xi)|. \tag{7.2}$$

Let $\mathfrak{A}^\varepsilon$ be the set of A^ε-valued progressively measurable processes $(\gamma_t)_{t \ge 0}$. Set $\bar{\mathfrak{A}}^\varepsilon = \cup_{n=1}^\infty \bar{\mathfrak{A}}^\varepsilon_n$, where $\bar{\mathfrak{A}}^\varepsilon_n$ is the set of progressively measurable processes $(\gamma_t)_{t \ge 0}$ with values in $\bar{A}^\varepsilon_n = A^\varepsilon \times [0,n]$.

Shaking the optimal reward w given by (2.7) means that we consider $\tilde{w} = \tilde{w}^\varepsilon(s,x)$ defined by

$$w^\varepsilon(s,x) = \sup_{\gamma \in \mathfrak{A}^\varepsilon} \sup_{\tau \in \mathfrak{T}(T-s)} w^{\gamma,\tau}(s,x),$$

where

$$w^{\gamma,\tau}(s,x) = \mathbb{E}^\gamma_{s,x}\left[\int_0^\tau f^{\gamma_t}(s+t, x_t)e^{-\varphi_t}\,dt + g^\varepsilon(s+\tau, x_\tau)e^{-\varphi_\tau}\right],$$

$$\varphi_t = \varphi_t^{\gamma,s,x} = \int_0^t c^{\gamma_r}(s+r, x_r^{\gamma,s,x})\,dr.$$

Notice that if Assumption 7.1 holds, then by virtue of Theorem 2.1

$$w^\varepsilon = \sup_{\gamma \in \bar{\mathfrak{A}}^\varepsilon} v^\gamma = \lim_{n\to\infty} w^\varepsilon_n,$$

where

$$v^\gamma(s,x) = \mathbb{E}^\gamma_{s,x}\left[\int_0^{T-s} f^{\gamma_t}(s+t, x_t)e^{-\varphi_t}\,dt + g^\varepsilon(T, x_{T-s})e^{-\varphi_{T-s}}\right],$$

$$w^\varepsilon_n = \sup_{\gamma \in \bar{\mathfrak{A}}^\varepsilon_n} v^\gamma. \tag{7.3}$$

Lemma 7.1. *Let Assumption 7.1 hold. Then there is a constant N such that*

$$|w^\varepsilon - w| \le N\varepsilon \quad \text{on } \bar{H}_T. \tag{7.4}$$

In addition to Assumption 7.1 assume that σ^α, β^α and f^α are continuous in $\alpha \in A$. Then there is a constant N such that

$$|w^\varepsilon(t,x) - w^\varepsilon(s,y)| \le N(|x-y| + |t-s|^{1/2}) \tag{7.5}$$

for $s,t \in [0,T]$, $x,y \in \mathbb{R}^d$. The constant N in the above estimates depends only on K and T. Moreover, there is a constant λ_0, depending only on K such that N is independent of T if $\lambda \ge \lambda_0$.

Proof. Applying Lemma 5.3 and Theorem 6.2 we immediately get estimate (7.5). By using the inequality

$$|a_1 e^{-b_1} - a_2 e^{-b_2}| \le |a_1 - a_2| + |a_1 + a_2||b_1 - b_2|,$$

for $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}_+$, for fixed $(s,x) \in \bar{H}_T$, $\alpha \in \mathfrak{A}$, $\tau \in \mathfrak{T}(T-s)$ and $\gamma = (\alpha, \eta, \xi) \in \mathfrak{A}^\varepsilon$ we have

$$|w^{\gamma,\tau}(s,x) - w^{\alpha,\tau}(s,x)| \le N_0(I_1 + I_2),$$

where

$$I_1 = \mathbb{E}\int_0^{T-s} e^{-\lambda t}(1+t)(|\varepsilon| + |x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|)\,dt$$

$$\le \varepsilon \int_0^T e^{-\lambda t}(t+1)\,dt + 3\mathbb{E}\sup_{t\le T-s} e^{-(\lambda-1)t}|x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|,$$

$$I_2 = \mathbb{E}e^{-\lambda\tau}(|\varepsilon| + |x_\tau^{\gamma,s,x} - x_\tau^{\alpha,s,x}|) + \mathbb{E}e^{-\lambda\tau}\int_0^\tau (|\varepsilon| + |x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|)\,dt$$

$$\le 2|\varepsilon| + 2\mathbb{E}\sup_{t\le T-s} e^{-(\lambda-1)t}|x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|,$$

and N_0 is a constant depending only on K. By Itô's formula we get

$$e^{-2(K^2+1)t}|x_t^{\gamma,s,x} - x_t^{\alpha,s,x}|^2 \le N_0\int_0^t e^{-2(K^2+1)r}\varepsilon^2\,dr + m_t \le N_1\varepsilon^2 + m_t,$$

where m is a local martingale and N_1 is a constant depending only on K. Hence

$$\mathbb{E}e^{-2(K^2+1)\rho}|x_\rho^{\gamma,s,x} - x_\rho^{\alpha,s,x}|^2 \le N_1\varepsilon^2$$

for stopping times ρ, that by virtue of Lemma 3.2 from [5] yields

$$\mathbb{E}\sup_{t\le T-s} e^{-(K^2+1)t}|x_t^{\gamma,s,x} - x_t^{\alpha,s,x}| \le 3\sqrt{N_1}|\varepsilon|.$$

Consequently, (7.4) holds with a constant N depending only on K and T, and if $\lambda \ge K^2+2$ then N is independent of T. □

Now we shake the finite difference problem (3.19)-(3.20) when $m^\alpha = 1$ for all $\alpha \in A$. We keep the notation of Section 3 and Assumption 3.1 in force. Moreover we make the following assumption.

Assumption 7.2. For $\psi := \sqrt{a_k^\alpha}, b_k^\alpha, f^\alpha, g, c^\alpha - \lambda$, for $\alpha \in A$ and $k = \pm 1, \dots, \pm d_1$ we have

$$|\psi(t,x)| \le K, \quad |\psi(t,x) - \psi(s,y)| \le K(|x-y| + |s-t|^{1/2})$$

for all $s,t \in \mathbb{R}$ and $x,y \in \mathbb{R}^d$.

Shaking the problem

$$\max[\sup_{\alpha\in A}\delta_\tau^T u + L_h^\alpha u + f^\alpha, g-u] = 0 \quad \text{on } H_T, \tag{7.6}$$

$$u(T,x) = g(T,x) \quad \text{for } x \in \mathbb{R}^d \tag{7.7}$$

means that we consider the problem

$$\max[\sup_{\gamma\in A^\varepsilon}\delta_\tau^T u + L_h^\gamma u + f^\gamma, g^\varepsilon - u] = 0 \tag{7.8}$$

$$u(T,x) = g^\varepsilon(T,x) \quad \text{for } x \in \mathbb{R}^d, \tag{7.9}$$

where g^ε is defined as in (7.2) and for $\gamma = (\alpha,\xi,\eta,r) \in \bar{A}^\varepsilon$

$$L_h^\gamma := a_k^\gamma\Delta_{h,\ell_k} + b_k^\gamma\delta_{h,\ell_k} - c^\gamma,$$

c^γ and f^γ are defined as in (7.1), and

$$a_k^\gamma(t,x) = a_k^\alpha(t+\eta, x+\xi), \quad b_k^\gamma(t,x) = b_k^\alpha(t+\eta, x+\xi), \quad t \in \mathbb{R}, x \in \mathbb{R}^d,$$

for $k = \pm 1, \dots, \pm d_1$.

By virtue of Theorem 3.4, if Assumptions 3.1 and 7.2 hold then (7.6)-(7.7) and (7.8)-(7.9) have a unique bounded solution $w_{\tau,h}$ and $w_{\tau,h}^\varepsilon$, respectively.

Lemma 7.2. *Let Assumptions 3.1 and 7.2 hold. Then*

$$|w^{\varepsilon}_{\tau,h} - w_{\tau,h}| \le N_0|\varepsilon| \quad on\ \bar{H}_T, \tag{7.10}$$

with a constant N_0 depending only on K, d_1 and T. Assume, additionally, $\tau, h \le K$. Then

$$|w^{\varepsilon}_{\tau,h}(s,x) - w^{\varepsilon}_{\tau,h}(t,y)| \le N_1(|x-y| + |s-t|^{1/2} + \sqrt{\tau}) \tag{7.11}$$

for all $s, t \in [0,T]$ and $x, y \in \mathbb{R}^d$, where N_1 is a constant depending only on K, d_1 and T. There is a constant λ_0 depending only on K and d_1 such that if $\lambda \ge \lambda_0$ then N_0 and N_1 are independent of T.

Proof. We get estimate (7.10) by an obvious application of Theorem 4.5. Estimate (7.11) follows immediately from Theorems 4.3 and 6.4. □

Let $\rho \in C_0^\infty(\mathbb{R}^{d+1})$ be a fixed nonnegative function with support in $(-1,0) \times B_1$ and unit integral, where B_1 denotes the open ball of radius 1 centered at the origin of $\mathbb{R}^d$. For $\varepsilon > 0$ set

$$w^{\varepsilon(\varepsilon)}(t,x) = \int_{\mathbb{R}^{d+1}} w^{\varepsilon}(s,y)\rho((t-s)/\varepsilon^2, (x-y)/\varepsilon)\, ds\, dy$$

for $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $w^{\varepsilon}(s,y) := w^{\varepsilon}(T,y)$ for $s \ge T$ and $y \in \mathbb{R}^d$. Define similarly $w^{\varepsilon(\varepsilon)}_{\tau,h}$ from $w^{\varepsilon}_{\tau,h}$.

Lemma 7.3. *Let Assumption 7.1 hold. Then there is a constant N_0 depending only on K and T such that*

$$|w^{\varepsilon(\varepsilon)} - w| \le N_0\varepsilon \quad on\ \bar{H}_T, \tag{7.12}$$

$$|w^{\varepsilon(\varepsilon)}(t,x) - w^{\varepsilon(\varepsilon)}(s,y)| \le N_0(|x-y| + |t-s|^{1/2}) \tag{7.13}$$

for all $s, t \in [0,T]$ and $x, y \in \mathbb{R}^d$. For integers $n \ge 1$

$$|D_t^n w^{\varepsilon(\varepsilon)}| + |D_x^{2n} w^{\varepsilon(\varepsilon)}| \le N_1 \varepsilon^{-2n+1} \quad on\ \bar{H}_T, \tag{7.14}$$

where N_1 is a constant depending only on n, K, d and T. There is a constant λ_0 such that if $\lambda \ge \lambda_0$ then N_0 and N_1 are independent of T. Moreover,

$$\max[D_t w^{\varepsilon(\varepsilon)} + \sup_{\alpha \in A}(L^\alpha w^{\varepsilon(\varepsilon)} + f^\alpha), g - w^{\varepsilon(\varepsilon)}] \le 0 \quad on\ H_T. \tag{7.15}$$

Proof. Estimates (7.12)-(7.14) follow immediately from Lemma 7.1. To prove (7.14) we use (7.3) and define $w^{\varepsilon(\varepsilon)}_n$ from w^{ε}_n as $w^{\varepsilon(\varepsilon)}$ is defined from w^{ε}. Notice that for $n \to \infty$

$$D_t w^{\varepsilon(\varepsilon)}_n \to w^{\varepsilon(\varepsilon)}, \quad D_x^\beta w^{\varepsilon(\varepsilon)}_n \to D_x^\beta w^{\varepsilon(\varepsilon)}_n$$

for multi-indices β, by Lebesgue's theorem on dominated convergence. By Theorem 2.1 in [13] for each integer $n \ge 1$ we have

$$D_t w^{\varepsilon(\varepsilon)}_n + L^\alpha w^{\varepsilon(\varepsilon)}_n + f^\alpha + r(g^\varepsilon - w^{\varepsilon(\varepsilon)}_n) \le 0 \quad \text{on } \bar{H}_T$$

for all $\alpha \in A$ and $r \in [0,1]$. Letting here $n \to \infty$ and using that $g \le g^\varepsilon$, we get

$$D_t w^{\varepsilon(\varepsilon)} + L^\alpha w^{\varepsilon(\varepsilon)} + f^\alpha + r(g - w^{\varepsilon(\varepsilon)}) \le 0 \quad \text{on } \bar{H}_T, \text{ for } \alpha \in A,\ r \ge 0,$$

that is equivalent to (7.14). □

Lemma 7.4. *Let Assumptions 3.1 and 7.2 hold. Then, provided $T > 2\varepsilon^2$,*

$$\max[\delta_\tau^T w_{\tau,h}^{\varepsilon(\varepsilon)} + \sup_{\alpha\in A}(L_h^\alpha w_{\tau,h}^{\varepsilon(\varepsilon)} + f^\alpha), g - w_{\tau,h}^{\varepsilon(\varepsilon)}] \leq 0 \quad \textit{on } H_{T-2\varepsilon^2}. \tag{7.16}$$

Assume, additionally, $\tau, h \leq K$. Then

$$|w_{\tau,h}^{\varepsilon(\varepsilon)} - w_{\tau,h}| \leq N_0(|\varepsilon| + \sqrt{\tau}) \quad \textit{on } \bar{H}_T, \tag{7.17}$$

$$|w_{\tau,h}^{\varepsilon(\varepsilon)}(t,x) - w_{\tau,h}^{\varepsilon(\varepsilon)}(s,y)| \leq N_0(|x-y| + |s-t|^{1/2} + \sqrt{\tau}), \tag{7.18}$$

for $t, s \in [0,T]$ and $x, y \in \mathbb{R}^d$, where N_0 is a constant depending only on K, d_1 and T. Moreover, for $n \geq 1$ there is a constant N_1 depending only on n, K, d_1, d and T, such that

$$|D_t^n w_{\tau,h}^{\varepsilon(\varepsilon)}| + |D_x^{2n} w_{\tau,h}^{\varepsilon(\varepsilon)}| \leq N_1 \varepsilon^{-2n}(|\varepsilon| + \sqrt{\tau}) \quad \textit{on } \bar{H}_T. \tag{7.19}$$

There is a constant λ_0 depending on K and d_1 such that if $\lambda \geq \lambda_0$ then N_0 and N_1 are independent of T.

Proof. Estimates (7.17)-(7.19) follow immediately from Lemma 7.2. To prove (7.16) notice that from (7.8) we have for $\alpha \in A$

$$(\delta_\tau^T + L_h^\alpha) w^{\varepsilon(\varepsilon)}(t - \varepsilon^2 s, x - \varepsilon y) + f^\alpha(t,x) \leq 0,$$

$$g(t,x) - w^{\varepsilon(\varepsilon)}(t - \varepsilon^2 s, x - \varepsilon y) \leq 0$$

for $(t,x) \in \bar{H}_{T-2\varepsilon^2}$, $s \in [-1,0]$, $|y| \leq 1$. Multiplying these inequalities by $\rho(s,y)$ and then integrating them against $ds\,dy$ we get (7.16). □

Proof of Theorem 2.4:

Let $\varepsilon = (\tau + h^2)^{1/4}$. Due to Theorems 6.2 and 6.4 it suffices to consider the case $T > 2\varepsilon^2$ and to prove (2.21) on H_S with $S = T - 2\varepsilon^2$. Notice that due to $\tau \leq 1$ we have $\tau < \varepsilon^2$. Hence for $u := w^{\varepsilon(\varepsilon)}, w_{\tau,h}^{\varepsilon(\varepsilon)}$ we have $\delta_\tau^T u = \delta_\tau u$ on H_S, and by Taylor's formula and using (7.14) and (7.19)

$$|\delta_\tau^T u - D_t u| + \sup_{\alpha\in A}|L^\alpha u - L_h^\alpha u|$$

$$\leq N_0(\tau \sup_{H_S}|D_t^2 u| + h^2 \sup_{H_S}|D_x^4 u| + h \sup_{H_S}|D_x^2 u|) \leq N_1 \varepsilon$$

on H_S. Notice also that

$$\sup_{H_T \setminus H_S}(w_{\tau,h} - w^{\varepsilon(\varepsilon)})_+$$

$$\leq \sup_{H_T\setminus H_S}(|w_{\tau,h} - g| + |g - w| + |w - w^{\varepsilon(\varepsilon)}|) \leq N_2 \varepsilon, \tag{7.20}$$

$$\sup_{\{S\}\times\mathbb{R}^d}(w - w_{\tau,h}^{\varepsilon(\varepsilon)})_+$$

$$\leq \sup_{\{S\}\times\mathbb{R}^d}(|w - g| + |g - w_{\tau,h}| + |w_{\tau,h} - w_{\tau,h}^{\varepsilon(\varepsilon)}|) \leq N_2 \varepsilon \tag{7.21}$$

Thus by (7.15) for $\alpha \in A$

$$\delta_\tau^T \bar{w}^{\varepsilon(\varepsilon)} + L_h^\alpha \bar{w}^{\varepsilon(\varepsilon)} + f^\alpha \leq 0 \quad \text{and} \quad g - \bar{w}^{\varepsilon(\varepsilon)} \leq 0 \;\text{ on } H_S \tag{7.22}$$

$$w_{\tau,h} \leq \bar{w}^{\varepsilon(\varepsilon)} \quad \text{on } \bar{H}_T \setminus H_S, \tag{7.23}$$

for $\bar{w}^{\varepsilon(\varepsilon)} := w^{\varepsilon(\varepsilon)} + N_1(S-t)\varepsilon + N_2\varepsilon$. If $\lambda > 0$ then (7.22) and (7.23) hold also for $\bar{w}^{\varepsilon(\varepsilon)} := w^{\varepsilon(\varepsilon)} + (N_1\lambda^{-1} + N_2)\varepsilon$. Similarly, by (7.16) for $\alpha \in A$

$$D_t\bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} + L^\alpha\bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} + f^\alpha \leq 0 \quad \text{and} \quad g - \bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} \leq 0 \quad \text{on } H_S \tag{7.24}$$

$$w \leq \bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} \quad \text{on } \{S\} \times \mathbb{R}^d \tag{7.25}$$

for $\bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} := w^{\varepsilon(\varepsilon)}_{\tau,h} + N_1(S-t)\varepsilon + N_2\varepsilon$ and also for $\bar{w}^{\varepsilon(\varepsilon)}_{\tau,h} := w^{\varepsilon(\varepsilon)}_{\tau,h} + (N_1\lambda^{-1} + N_2)\varepsilon$ when $\lambda > 0$. By Corollary 3.10 from (7.22)-(7.23) we get $w_{\tau,h} \leq \bar{w}^{\varepsilon(\varepsilon)}$, and by Lemma 5.4 from (7.24)-(7.25) we have $w \leq \bar{w}^{\varepsilon(\varepsilon)}_{\tau,h}$ on H_S. Consequently, there is a constant N such that

$$w_{\tau,h} \leq w + N\varepsilon, \quad w \leq w_{\tau,h} + N\varepsilon \quad \text{on } H_S,$$

that obviously yields (2.21) on H_S. Inspecting the constants N_0, N_1 and N_2 we see that N depends only on K, d, d_1 and T, and that there is a constant λ_0, depending only on K and d_1 such that if $\lambda \geq \lambda_0$ then N is independent of T.

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